

A Second-Order Logic Primer*

Robert Trueman

1 Introducing Second-Order Logic

Here is a good, natural language argument:

Bertrand is a logician

Bertrand is a mathematician

\therefore Someone is both a logician and a mathematician

Here is another good, natural language argument:

Bertrand is a philosopher

Alfred is a philosopher

\therefore Bertrand and Alfred have something in common

But while both of these are perfectly good arguments, the formal tools you have been given deal only with the first, not the second. As you all know, you can formalise the first argument like this:

$$Lb, Mb \therefore \exists x(Lx \wedge Mx)$$

What is more, you all also know how to construct a natural deduction proof which vindicates this argument. But how would you go about formalising the second of our two arguments? And once you had formalised it, how would you construct a proof to vindicate it? There is no way for you to answer these questions with the formal tools you currently have access to.

What we need to do is introduce a new kind of variable. The variables in first-order logic (FOL) all go where *names* go. We need some new variables, which go where *predicates* go. If we had variables like these, we could formalise our second argument like this:

*When preparing this primer, I found Button and Walsh's *Philosophy and Model Theory* (2018, OUP) very helpful.

$$Pb, Pa \therefore \exists X(Xb \wedge Xa)$$

When we add these new variables, we take the step from first-order logic to *second-order logic* (SOL). Here's a rough way to think about the difference between FOL and SOL. The quantifiers in FOL let us quantify over *objects*: ' $\exists x(Lx \wedge Mx)$ ' says that there is some *object* which is both L and M . The new quantifiers in SOL, on the other hand, let us quantify over *properties*: ' $\exists X(Xb \wedge Xa)$ ' says that there is some *property* which b and a both have.

This is a very rough and ready characterisation of SOL. Our aim in this little primer is to get clearer on the details. We will start by describing the language of SOL (§2). Then we will go over the natural deduction rules for SOL (§3). After that, we will look at the semantics for SOL (§4). And then we will end by briefly looking at the second-order definition of identity (§5).

2 The Language of SOL

SOL is an extension of FOL. To get the basic vocabulary of SOL, all we have to do is take FOL, and add new variables which go in the place of predicates.¹ We will use capital letters from S to Z as these new variables, which we will call *second-order variables*. (By contrast, the lowercase variables from FOL will now be called *first-order variables*.)

Here is a complete list of all the basic symbols of SOL:

Names with subscripts, as needed	a, b, c, \dots, r $a_1, b_{224}, h_7, m_{32}, \dots$
Predicates (with superscripts) and with subscripts, as needed	$=, A^1, B^1, \dots, R^1, A^2, B^2, \dots$ $A_1^1, B_1^2, R_1^5, A_2^8, J_{25}^{10}, \dots$
First-order variables with subscripts, as needed	s, t, u, v, w, x, y, z $x_1, y_1, z_1, x_2, \dots$
Second-order variables with subscripts, as needed	S, T, U, V, W, X, Y, Z $X_1, Y_1, Z_1, X_2, \dots$
Connectives	$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
Brackets	$(,)$
Quantifiers	\forall, \exists

¹That is not actually quite right. Really we are going to take some of the letters that we were using as predicates in FOL, and re-purpose them as variables.

The superscripts on the predicates indicate their ‘adicity’: a *monadic* predicate is a predicate which combines with one term at a time; a *dyadic* predicate is a predicate which combines with two terms at a time; and in general, an *n-adic* predicate is a predicate which combines with n terms at a time. (For ease of expression, we often won’t bother to actually write these superscripts in.) Second-order variables can also have any adicity, but to keep things simple, all of our second-order variables are monadic: one of our second-order variables can only combine with a single term at a time.² To be absolutely clear, we really are doing this just to keep things simple. There is absolutely no technical reason not to include second-order variables of higher adicities.

Now that we have a basic vocabulary for SOL, we need to explain how to build sentences out of it. We will do this in three steps, just as we did for FOL: first we will explain how to build *atomic formulas*; then we will explain how to build more complex formulas from simpler ones; and finally, we will explain which formulas count as *sentences*.

We start with our definition of a *term* from FOL:

A TERM is any name or first-order variable.

With this definition to hand, we can define the *atomic formulas* of SOL. This definition is exactly the same as it was for FOL, but with one extra clause, clause 3:

1. If \mathcal{R}^n is an n -place predicate and t_1, t_2, \dots, t_n are terms, then $\mathcal{R}^n t_1 t_2 \dots t_n$ is an atomic formula.
2. If t_1 and t_2 are terms, then $t_1 = t_2$ is an atomic formula.
3. If \mathcal{X} is a second-order variable and t is a term, then $\mathcal{X}t$ is an atomic formula.
4. Nothing else is an atomic formula.

Here are some examples of atomic SOL formulas:

Fa, Fx, Xa, Yx

Now it is time to give the rules for building more complex formulas from simpler ones. These rules are all the same as they were in FOL, apart from rule 8:

²Logicians call this simplified version of SOL: *monadic second-order logic*.

1. Every atomic formula is a formula.
2. If \mathcal{A} is a formula, then $\neg\mathcal{A}$ is a formula.
3. If \mathcal{A} and \mathcal{B} are formulas, then $(\mathcal{A} \wedge \mathcal{B})$ is a formula.
4. If \mathcal{A} and \mathcal{B} are formulas, then $(\mathcal{A} \vee \mathcal{B})$ is a formula.
5. If \mathcal{A} and \mathcal{B} are formulas, then $(\mathcal{A} \rightarrow \mathcal{B})$ is a formula.
6. If \mathcal{A} and \mathcal{B} are formulas, then $(\mathcal{A} \leftrightarrow \mathcal{B})$ is a formula.
7. If \mathcal{A} is a formula, χ is a first-order variable, \mathcal{A} contains at least one occurrence of χ , and \mathcal{A} contains neither $\forall\chi$ nor $\exists\chi$, then $\forall\chi\mathcal{A}$ and $\exists\chi\mathcal{A}$ are both formulas.
8. If \mathcal{A} is a formula, \mathcal{X} is a second-order variable, \mathcal{A} contains at least one occurrence of \mathcal{X} , and \mathcal{A} contains neither $\forall\mathcal{X}$ nor $\exists\mathcal{X}$, then $\forall\mathcal{X}\mathcal{A}$ and $\exists\mathcal{X}\mathcal{A}$ are formulas.
9. Nothing else is a formula.

Here are some examples of SOL formulas:

$$Fa, Fx, Xa, \exists x(Fx \rightarrow Gx), \exists X(Xa \rightarrow Ga), \forall x\exists Y(Yx \leftrightarrow Za)$$

Now it is time for the final step. We define the *sentences* of SOL as follows:

A SENTENCE of SOL is any formula of SOL which contains no free variables (first-order or second-order).

The definition of a free variable is just the same whether we are dealing with first-order variables or second-order ones: a first-order variable, χ , is free iff it is not within the scope of either $\forall\chi$ or $\exists\chi$; a second-order variable, \mathcal{X} , is free iff it is not within the scope of either $\forall\mathcal{X}$ or $\exists\mathcal{X}$. Here are some examples of sentences of SOL:

$$Fa, \exists xFx, \exists X Xa, \exists x(Fx \rightarrow Gx), \exists X(Xa \rightarrow Ga), \forall x\exists Y\forall Z(Yx \leftrightarrow Za)$$

3 Natural Deduction for SOL

Now that we know how to make sentences in SOL, we can look at how to *prove* things in SOL. Just as in *forall* χ , we will use \vdash to express provability. However, to make it clear that we are using SOL, not FOL, we will add a subscripted 2. So if we want to say that we can prove \mathcal{C} from $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ in SOL, we will write: $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \vdash_2 \mathcal{C}$.

3.1 Introduction and Elimination Rules

The natural deduction system for SOL is an extension of the natural deduction system for FOL. So SOL includes *all* of the rules of FOL (basic and derived). All we need to do now is add some extra rules to govern *second-order quantifiers* (i.e. quantifiers which bind second-order variables). First and foremost, we need to give them introduction and elimination rules. Happily, we can go through these quite quickly, because they are almost exactly the same as the rules for the *first-order quantifiers* (i.e. quantifiers which bind first-order variables).

We start with *Second-Order Existential Introduction*:

$ \begin{array}{c l} m & \mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots) \\ & \exists \mathcal{X} \mathcal{A}(\dots \mathcal{X} \dots \mathcal{F} \dots) \quad \exists_2\text{I}, m \\ \hline & \mathcal{X} \text{ must not occur in } \mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots) \end{array} $

Here is how to understand the notation in this rule: \mathcal{F} is a one-place predicate; \mathcal{X} is a second-order variable; $\mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots)$ is a formula containing one or more occurrence of \mathcal{F} ; and $\mathcal{A}(\dots \mathcal{X} \dots \mathcal{F} \dots)$ is a formula obtained by replacing *some* or *all* of those occurrences of \mathcal{F} with occurrences of \mathcal{X} .

We turn now to *Second-Order Existential Elimination*:

$ \begin{array}{c l} m & \exists \mathcal{X} \mathcal{A}(\dots \mathcal{X} \dots \mathcal{X} \dots) \\ i & \left \begin{array}{l} \mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots) \\ \hline \mathcal{B} \end{array} \right. \\ j & \left \mathcal{B} \right. \\ & \mathcal{B} \quad \exists_2\text{E}, m, i-j \\ \hline & \mathcal{F} \text{ must not occur in any assumption undischarged before line } i \\ & \mathcal{F} \text{ must not occur in } \exists \mathcal{X} \mathcal{A}(\dots \mathcal{X} \dots \mathcal{X} \dots) \\ & \mathcal{F} \text{ must not occur in } \mathcal{B} \end{array} $

A quick reminder of notation: $\mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots)$ is the result of substituting \mathcal{F} for *all* of the occurrences of \mathcal{X} in $\mathcal{A}(\dots \mathcal{X} \dots \mathcal{X} \dots)$.

Now that we are finished with the existential quantifier, it is time to give the rules for the universal quantifier. We start with *Second-Order Universal Introduction*:

$m \quad \left \begin{array}{l} \mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots) \\ \forall \mathcal{X} \mathcal{A}(\dots \mathcal{X} \dots \mathcal{X} \dots) \end{array} \right. \quad \forall_2 \text{I}, m$ <p>\mathcal{F} must not occur in any undischarged assumption \mathcal{F} must not occur in $\forall \mathcal{X} \mathcal{A}(\dots \mathcal{X} \dots \mathcal{X} \dots)$ \mathcal{X} must not occur in $\mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots)$</p>
--

And here is *Second-Order Universal Elimination*:

$m \quad \left \begin{array}{l} \forall \mathcal{X} \mathcal{A}(\dots \mathcal{X} \dots \mathcal{X} \dots) \\ \mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots) \end{array} \right. \quad \forall_2 \text{E}, m$
--

The best way to get a feel for these rules is to actually use them in some proofs. So here are some exercises for you to try. Solutions are in §6.

Provide proofs for the following:

1. $Pa, Pb \vdash_2 \exists X(Xa \wedge Xb)$
2. $Fa, \neg Fb \vdash_2 \neg \forall Y(Ya \leftrightarrow Yb)$
3. $\exists X(Xa \wedge \neg Xb) \vdash_2 \neg a = b$
4. $\exists X \exists y \neg Xy \vdash_2 \neg \forall X \forall y Xy$
5. $\neg \exists x \neg x = a \vdash_2 \forall X(Xa \rightarrow \forall x Xx)$
6. $\forall X(Xa \rightarrow \forall y(\neg y = a \rightarrow \neg Xy)), Qa \vdash_2 \exists Z \neg \exists x \exists y(\neg x = y \wedge (Zx \wedge Zy))$

3.2 Comprehension

Our natural deduction system for SOL already lets us prove lots of things, but it does have its limits. Consider the following argument:

Susanne is a pianist or an historian

Mary is a pianist or an historian

\therefore Susanne and Mary have something in common

This strikes me as a good argument. Even if Susanne isn't a historian and Mary isn't a pianist, they still have *something* in common: they are both pianists or historians! We can symbolise this argument in SOL as follows:

$$Ps \vee Hs, Pm \vee Hm \therefore \exists X(Xs \wedge Xm)$$

Unfortunately, however, the rules we have laid out so far will *not* allow us to provide a proof to vindicate this argument. The trouble is that these rules only ever allow us to replace simple predicates, like ‘ P ’ and ‘ H ’, with second-order variables, not complex predicates like ‘ $Px \vee Hx$ ’. As far as the rules we currently have are concerned, it is only simple predicates which define properties, not complex ones.

To get around this problem, we need to add a rule which allows us to define properties with complex predicates. That rule is known as *Comprehension*:

$$\begin{array}{l} | \exists X \forall \chi (X\chi \leftrightarrow \mathcal{A}(\dots \chi \dots \chi \dots)) \quad \text{Comp} \\ \mathcal{X} \text{ must not occur in } \mathcal{A}(\dots \chi \dots \chi \dots) \end{array}$$

This rule essentially allows us to stop at any point in a proof, and define a new property with the formula $\mathcal{A}(\dots \chi \dots \chi \dots)$. The best way to see how this new rule is meant to work is by seeing it in action. Here is a proof vindicating the argument we were discussing just a moment ago:

1	$Ps \vee Hs$															
2	$Pm \vee Hm$															
3	$\exists X \forall x (Xx \leftrightarrow (Px \vee Hx))$	Comp														
4	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$\forall x (Fx \leftrightarrow (Px \vee Hx))$</td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$Fs \leftrightarrow (Ps \vee Hs)$</td> <td>$\forall_1 E, 4$</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">Fs</td> <td>$\leftrightarrow E, 5, 1$</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$Fm \leftrightarrow (Pm \vee Hm)$</td> <td>$\forall_1 E, 4$</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">Fm</td> <td>$\leftrightarrow E, 7$</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$Fs \wedge Fm$</td> <td>$\wedge I, 6, 8$</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$\exists X (Xs \wedge Xm)$</td> <td>$\exists_2 I, 9$</td> </tr> </table>	$\forall x (Fx \leftrightarrow (Px \vee Hx))$		$Fs \leftrightarrow (Ps \vee Hs)$	$\forall_1 E, 4$	Fs	$\leftrightarrow E, 5, 1$	$Fm \leftrightarrow (Pm \vee Hm)$	$\forall_1 E, 4$	Fm	$\leftrightarrow E, 7$	$Fs \wedge Fm$	$\wedge I, 6, 8$	$\exists X (Xs \wedge Xm)$	$\exists_2 I, 9$	
$\forall x (Fx \leftrightarrow (Px \vee Hx))$																
$Fs \leftrightarrow (Ps \vee Hs)$	$\forall_1 E, 4$															
Fs	$\leftrightarrow E, 5, 1$															
$Fm \leftrightarrow (Pm \vee Hm)$	$\forall_1 E, 4$															
Fm	$\leftrightarrow E, 7$															
$Fs \wedge Fm$	$\wedge I, 6, 8$															
$\exists X (Xs \wedge Xm)$	$\exists_2 I, 9$															
11	$\exists X (Xs \wedge Xm)$	$\exists_2 E, 3, 4-10$														

Importantly, the formula you plug in for $\mathcal{A}(\dots \chi \dots \chi \dots)$ can be as complex as you like. It can contain first-order quantifiers, if you want. It can even contain *second-order* quantifiers.³ \mathcal{A} can be *any* formula, so long as it contains χ but doesn't

³Some logicians restrict Comprehension so that second-order quantifiers cannot appear in \mathcal{A} .

contain \mathcal{X} .

And there we have it, that's our natural deduction system for SOL! It includes all of the rules of FOL, plus the introduction and elimination rules for the second-order quantifiers, *plus* Comprehension.⁴ If you want to get a better handle on how this system works, then try the following exercises (solutions in §6):

Provide proofs for the following:

1. $Aa, \neg Ab \vdash_2 \exists Y(\neg Ya \wedge Yb)$
2. $\vdash_2 \exists X \forall x \neg Xx$
3. $\exists Y \forall X \forall z (Yz \leftrightarrow Xz) \vdash_2 \perp$
4. $\vdash_2 \forall Z (\forall Y \forall x (Yx \rightarrow Zx) \rightarrow \forall x Zx)$
5. $\forall X ((Xa \wedge Xb) \rightarrow Xc), \forall x \forall y (Rxy \rightarrow Ryx), \exists x Rax \wedge \exists x Rbx \vdash_2 \exists y Ryc$

4 Semantics for SOL

So far, we have focussed on the natural deduction system for SOL. Now we will look at the *semantics* for SOL. A semantics for a language is a method for assigning truth-values to the sentences in that language. So a semantics for SOL is a method for assigning truth-values to the sentences of SOL.

In particular, we are going to focus on what is known as the *Standard Semantics* for SOL. This is not the only semantics — there are others, e.g. *Henkin Semantics* — but it is the especially philosophically interesting. This Standard Semantics is fundamentally *set-theoretical*, and so we will need to spend a moment going over some of the basics of set-theory.

4.1 Set-Theory: The Basics

The aim in this subsection is just to introduce you to enough set-theory to understand the Standard Semantics for SOL. This is *absolutely not* a set-theory textbook! If you want a textbook that will introduce you to modern set-theory (known as *ZFC*), then I strongly recommend Goldrei's *Classic Set Theory* (published in 1996 by Chapman & Hall).

Sets are collections of objects. For example, the set of humans is a collection containing every human (and nothing else). We can put anything we like in a set.

When we restrict Comprehension in this way, it is known as *Predicative Comprehension*.

⁴Logicians often add another rule, known as *Choice*, but adding that would force us to move beyond monadic SOL.

If we like, these things can have some natural property in common, but they don't have to. For example, there is a set which contains our Sun, the 'T' on my keyboard, the Eiffel Tower, and nothing else. This is a totally miscellaneous set of things, but it is still a set.

Mathematicians use curly brackets to refer to sets. For example, {the Sun, my 'T' key, the Eiffel Tower} is the set containing the Sun, my 'T' key, the Eiffel Tower and nothing else. In this example, we have listed the things in the set, but we can also refer to the set of F s like this: $\{x : Fx\}$. For example, if we wanted to refer to the set of dogs without having to actually list all the dogs, we could refer to it as: $\{x : x \text{ is a dog}\}$. In fact, this is the fundamental way of using the set brackets, as we can always re-write $\{a_1, a_2, \dots, a_n\}$ like this: $\{x : x = a_1 \vee x = a_2 \vee \dots \vee x = a_n\}$.

The things which belong in a set are called its *members* (or sometimes its *elements*). We use \in to express the membership relation. For example, if we want to say that Fido is a member of the set of dogs, we write: $\text{Fido} \in \{x : x \text{ is a dog}\}$. One of the most important facts about sets is that they are *extensional*. This means that set a is identical to set b iff a and b have exactly the same members. Or in symbols: $a = b \leftrightarrow \forall x(x \in a \leftrightarrow x \in b)$.

One set can be a *subset* of another, which we write as follows: $a \subseteq b$. This means that every member of a is a member of b . Or in symbols: $\forall x(x \in a \rightarrow x \in b)$. There is one special set which is a subset of *every set*. It is the *empty set*, which has no members at all. We could define the empty set as $\{x : \neg x = x\}$, but most mathematicians just write it as \emptyset . I'll leave it as an exercise for you to prove that \emptyset is a subset of every set!

Sets can have other sets as members. Consider the following two sets: {Kant, Frege} and {Russell, Wittgenstein}. If we want, we can form a set of those sets: {{Kant, Frege}, {Russell, Wittgenstein}}. And we don't have to stop there: we can make sets of sets of sets, and sets of sets of sets of sets, and so on forever. We do have to be a little careful at this point, otherwise we will run into Russell's Paradox. However, modern set-theory, called *ZFC*, let's us take sets of sets of sets... without landing ourselves in paradox. We cannot go into all of the details here, but the guiding idea is this: you can always build a new set out of an old one by forming the set of all the *subsets* of the old set. This is known as the *Powerset Operation*, and the powerset of a is the set of all the subsets of a : $\mathcal{P}(a) = \{x : x \subseteq a\}$.

Sets are *unordered*, in the following sense: $\{a, b\} = \{b, a\}$. You can deduce that sets must be unordered from the fact that they are extensional: $\{a, b\}$ and $\{b, a\}$ have exactly the same members, a and b . However, we can also work with collections which are ordered. We write the *ordered pair* of a and b like this: $\langle a, b \rangle$. Ordered

pairs are ordered in the following sense: $\langle a, b \rangle = \langle c, d \rangle \leftrightarrow (a = c \wedge b = d)$. It turns out that we can build ordered pairs out of unordered sets as follows: $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$. I will leave it as an exercise for you to show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ iff $a = c$ and $b = d$.

What if we want to deal with bigger ordered sets? What if we want an ordered *triple*, not just an ordered pair? Well, we can build them out of ordered pairs: $\langle a, b, c \rangle = \langle \langle a, b \rangle, c \rangle$. And we can then build bigger and bigger ordered sequences in a similar way.

At various points in mathematics it can be useful to consider the *Cartesian Product* of two sets, a and b , which we write as: $a \times b$. This is the set of all the ordered pairs whose first member is a member of a , and whose second is a member of b . In symbols: $a \times b = \{\langle x, y \rangle : x \in a \wedge y \in b\}$. For example, $\{\text{Kant}\} \times \{\text{Russell}, \text{Wittgenstein}\} = \{\langle \text{Kant}, \text{Russell} \rangle, \langle \text{Kant}, \text{Wittgenstein} \rangle\}$. We can also take the Cartesian Product of a set with itself: $a \times a = a^2$. This is the set of all ordered pairs you can make out of the members of a . We can also construct $a^2 \times a = a^3$, which is the set of all ordered triples of members of a , and so on. In general, we have the following rule: $a^1 = a$,⁵ and $a^{n+1} = a^n \times a$.

That is enough set-theory! Although we have barely scratched the surface of what set-theory has to offer, you have now learnt enough to understand the Standard Semantics for SOL.

4.2 The Standard Semantics

The Standard Semantics is really just a simple extension of the semantics that you already learnt for FOL. But before we get to that extension, I want to go back over the basics. Most textbooks use lots of set-theoretic notions in their semantics for FOL. I made a point of avoiding that last term, because we didn't really need any set-theory for what we were doing. But now that you are familiar with some of the set-theoretic fundamentals, it is worth taking a moment to re-do our semantics for FOL in set-theoretic terms.

We start with the idea of an *interpretation*. From now on, you should think of a domain as a *set* of objects. You should also think of the extension of a monadic predicate as a *subset* of the domain, the extension of a dyadic predicate is a set of ordered pairs of members of the domain, and so on. Formally, we can define an interpretation as follows:

⁵This base clause requires that we identify each object x with its 'ordered single' $\langle x \rangle$. That's fine, but don't let it confuse you: x is never identical to $\{x\}$! This of course implies that $\{x\} \neq \langle x \rangle$.

An INTERPRETATION is an ordered pair, $\langle d, \nu \rangle$, where d is a set of objects and ν is a valuation function; ν maps names to objects in d , and n -adic predicates to subsets of d^n .

Now that we have defined what an interpretation is, we can give a recursive definition of *truth in an interpretation*. We start with atomic sentences.

Let \mathcal{R}^n be any n -place predicate, and $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ be any names:
 $\mathcal{R}^n \mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n$ is true in $\langle d, \nu \rangle$ **iff** $\langle \nu(\mathbf{a}_1), \nu(\mathbf{a}_2), \dots, \nu(\mathbf{a}_n) \rangle \in \nu(\mathcal{R}^n)$
 $\mathbf{a}_1 = \mathbf{a}_2$ is true in $\langle d, \nu \rangle$ **iff** $\nu(\mathbf{a}_1) = \nu(\mathbf{a}_2)$

Now it is time for the truth-functional connectives:

$\mathcal{A} \wedge \mathcal{B}$ is true in $\langle d, \nu \rangle$ **iff** \mathcal{A} is true in $\langle d, \nu \rangle$ and \mathcal{B} is true in $\langle d, \nu \rangle$
 $\mathcal{A} \vee \mathcal{B}$ is true in $\langle d, \nu \rangle$ **iff** \mathcal{A} is true in $\langle d, \nu \rangle$ or \mathcal{B} is true in $\langle d, \nu \rangle$
 $\neg \mathcal{A}$ is true in $\langle d, \nu \rangle$ **iff** \mathcal{A} is not true in $\langle d, \nu \rangle$
 $\mathcal{A} \rightarrow \mathcal{B}$ is true in $\langle d, \nu \rangle$ **iff** \mathcal{A} is not true in $\langle d, \nu \rangle$ or \mathcal{B} is true in $\langle d, \nu \rangle$
 $\mathcal{A} \leftrightarrow \mathcal{B}$ is true in $\langle d, \nu \rangle$ **iff** \mathcal{A} and \mathcal{B} have the same truth-value in $\langle d, \nu \rangle$

Next up are the first-order quantifiers. Now, in *forall* χ , we dealt with first-order quantifiers by *adding names* into our language. This is a very intuitive idea, but from a formal perspective, it is irritating to add or remove names in the course of our semantics. It is actually better to just pick a name which you haven't used yet, as follows:

Let \mathbf{c} be any name which does not appear in $\mathcal{A}(\dots \chi \dots \chi \dots)$.
 $\forall \chi \mathcal{A}(\dots \chi \dots \chi \dots)$ is true in $\langle d, \nu \rangle$ **iff**
 $\mathcal{A}(\dots \mathbf{c} \dots \mathbf{c} \dots)$ is true in *every* interpretation that differs from $\langle d, \nu \rangle$ only by mapping \mathbf{c} to a different member of d , if it differs at all.
 $\exists \chi \mathcal{A}(\dots \chi \dots \chi \dots)$ is true in $\langle d, \nu \rangle$ **iff**
 $\mathcal{A}(\dots \mathbf{c} \dots \mathbf{c} \dots)$ is true in *some* interpretation that differs from $\langle d, \nu \rangle$ only by mapping \mathbf{c} to a different member of d , if it differs at all.

To be absolutely clear, this is essentially the same semantic clause we gave in *forall* χ . The only difference is that we are now thinking of interpretations set-theoretically, and we have avoided adding new names into the language.

We are at long last ready to give the semantic clauses for the second-order quantifiers:

Let \mathcal{F} be any one-place predicate which does not appear in $\mathcal{A}(\dots\mathcal{X}\dots\mathcal{X}\dots)$.

$\forall\mathcal{X}\mathcal{A}(\dots\mathcal{X}\dots\mathcal{X}\dots)$ is true in $\langle d, \nu \rangle$ **iff**
 $\mathcal{A}(\dots\mathcal{F}\dots\mathcal{F}\dots)$ is true in *every* interpretation that differs from $\langle d, \nu \rangle$ only by mapping \mathcal{F} to a different subset of d , if it differs at all.

$\exists\mathcal{X}\mathcal{A}(\dots\mathcal{X}\dots\mathcal{X}\dots)$ is true in $\langle d, \nu \rangle$ **iff**
 $\mathcal{A}(\dots\mathcal{F}\dots\mathcal{F}\dots)$ is true in *some* interpretation that differs from $\langle d, \nu \rangle$ only by mapping \mathcal{F} to a different subset of d , if it differs at all.

I hope that these clauses look reassuringly familiar to the clauses for the first-order quantifiers. As ever, the best way to understand how these clauses work is to look at some examples of them in action. Imagine we had an interpretation which looked like this:

$$d = \{\text{Frege, Russell}\}$$

$$\nu('a') = \text{Frege}$$

$$\nu('b') = \text{Russell}$$

$$\nu('A') = \{\text{Frege}\}$$

$$\nu('B') = \{\text{Russell}\}$$

What truth-values do the following sentences get on this interpretation?

1. $\exists\mathcal{X}(Xa \wedge \neg Xb)$
2. $\forall\mathcal{X}(Xa \leftrightarrow \neg Xb)$
3. $\exists\mathcal{X}\forall x(Xx \leftrightarrow (Ax \vee Bx))$

Let's go through these sentences one at a time, starting with 1. If we want to know whether 1 is true, we need to delete the quantifier, and replace all the X s

with some predicate which we haven't used yet. This will do: ' $Ca \wedge \neg Cb$ '. Now we must ask: Is this new sentence true on some interpretation which differs from our original interpretation at most over what extension it gives ' C '? The answer to this question is Yes: just imagine an interpretation in which $\nu('C') = \{\text{Frege}\}$. And since the answer is Yes, 1 must be true on our original interpretation.

Now let's look at 2. To decide whether 2 is true, we again need to delete the quantifier and replace the X s with some predicate which doesn't appear in this sentence. This will do: ' $Ca \leftrightarrow \neg Cb$ '. Now we must ask: Is this new sentence true on every interpretation which differs from our original interpretation at most over what extension it gives ' C '? The answer to this question is No: just imagine an interpretation in which $\nu('C') = \emptyset$. And since the answer is No, 2 must also be false on our original interpretation.

And finally, it is time for 3. To decide whether 3 is true, we need to ask whether ' $\forall x(Cx \leftrightarrow (Ax \vee Bx))$ ' is true on some interpretation which differs from our original interpretation at most over what extension it gives ' C '. And there is such an interpretation: just let $\nu('C') = \{\text{Frege, Russell}\}$. So 3 is true on our original interpretation.

4.3 Logical Consequence

Now that we have a workable semantics for SOL, we can define *logical consequence* for SOL too:

$\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models_2 C$ iff every interpretation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ true also makes C true.

Whenever you are introduced to a semantic concept of logical consequence (expressed by \models), you should always ask how it connects to the syntactic concept of provability (expressed by \vdash). There are two important possible relations, *soundness* and *completeness*:

- **Soundness:** if $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \vdash_2 C$ then $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models_2 C$
- **Completeness:** if $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models_2 C$ then $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \vdash_2 C$

Now, *every* logic we have looked at so far has been sound and complete. But not SOL. The natural deduction system I have laid out is sound. (Thank goodness!) But it isn't complete. And that isn't just because I accidentally missed out some rules. It turns out that no system of natural deduction for SOL can be both sound and complete!

Why is that? Well, although this might sound weird, SOL is too *powerful* to be captured in a single system of natural deduction. Unfortunately, proving this amazing result is well beyond our means here. The first step in the proof is Gödel's Incompleteness Theorems! But if you are interested in this result, then you should take the *Foundations of Mathematics* module next year. You won't quite learn how to prove Gödel's theorems in all their glory, but you will get a sense of what they mean, and why they are so important!

5 Identity in SOL

I want to end this little primer with a brief discussion of identity. In FOL, identity is undefinable. You have to take it as a primitive, basic logical concept. But in SOL, we can *define identity*!

$$\boxed{\forall x \forall y (x = y \leftrightarrow_{df} \forall X (Xx \leftrightarrow Xy))}$$

In English, the idea is that x is identical to y iff x and y have *exactly the same properties*. This is an old definition of identity, which goes back to Leibniz. Leibniz proposed two principles:

- **Indiscernibility of Identicals:** $\forall x \forall y (x = y \rightarrow \forall X (Xx \leftrightarrow Xy))$
- **Identity of Indiscernibles:** $\forall x \forall y (\forall X (Xx \leftrightarrow Xy) \rightarrow x = y)$

The second-order definition of identity is what you get when you put these two principles together.

Now, if you've done much metaphysics, then you will probably have heard people say that while the Indiscernibility of Identicals is incontrovertible,⁶ the Identity of Indiscernibles is very dubious. However, it turns out to be fairly easy to show that the Identity of Indiscernibles is true in every interpretation on the Standard Semantics. Suppose ' a ' and ' b ' refer to two distinct objects, 1 and 2. Now consider this sentence: ' $Aa \leftrightarrow Ab$ '. It would be easy to find an extension for ' A ' which would make this sentence false. We could let the extension of ' A ' be $\{1\}$, or we could let it be $\{2\}$. Either way, ' $Aa \leftrightarrow Ab$ ' will come out false. As a result, ' $\forall X (Xa \leftrightarrow Xb)$ ' must be false on our interpretation. Thus, ' $\forall X (Xa \leftrightarrow Xb)$ ' can only be true on an interpretation if ' a ' and ' b ' refer to the same thing. But ' a ' and ' b ' were arbitrarily chosen names. Thus ' $\forall x \forall y (\forall X (Xx \leftrightarrow Xy) \rightarrow x = y)$ ' is true on every interpretation.

⁶In fact, it is often known as *Leibniz's Law*!

So, does that show that there is absolutely nothing wrong with the Identity of Indiscernibles, and that all of the metaphysical toing and froing was a waste of time? *Not at all!* When I first introduced you to second-order logic, I told you that in second-order logic we can quantify over *properties* as well as objects. But in our official Standard Semantics, we ended up swapping properties for *sets*. Now, there is nothing at all wrong with this. Sets are well behaved mathematical objects, and so for formal purposes, they are much better to work with than properties. But for metaphysical purposes, it is properties which matter most. A metaphysician should, then, think of the set-theoretic Standard Semantics as a mere model of what they are really interested in. The sets that we assign to predicates merely *represent* the properties we really care about. When we look at the Standard Semantics like that, we have to ask ourselves: How well does our set-theoretic model represent reality? And at this point, it becomes *very* interesting to ask whether it is possible for two distinct objects to share all of their properties.

Further Reading

One of the standard textbooks for SOL is:

Shapiro, S. (1991) *Foundations without Foundationalism: A Case for Second-Order Logic*, Oxford University Press.

If you are interested in the philosophy behind SOL, then (a little bit cheekily), I would suggest:

Trueman, R. (2021) *Properties and Propositions: The Metaphysics of Higher-Order Logic*, Cambridge University Press.

And if you would like to know more about set-theory, I recommend:

Goldrei, D. (1996) *Classic Set Theory*, Chapman & Hall.

6 Solutions

Solutions for §3.1

Provide proofs for the following:

1. $Pa, Pb \vdash_2 \exists X(Xa \wedge Xb)$

1	Pa	
2	Pb	
3	$Pa \wedge Pb$	$\wedge I, 1, 2$
4	$\exists X(Xa \wedge Xb)$	$\exists_2 I, 3$

2. $Fa, \neg Fb \vdash_2 \neg \forall Y(Ya \leftrightarrow Yb)$

1	Fa	
2	$\neg Fb$	
3	$\forall Y(Ya \leftrightarrow Yb)$	
4	$Fa \leftrightarrow Fb$	$\forall_2 E, 3$
5	Fb	$\leftrightarrow E, 4, 1$
6	\perp	$\perp I, 5, 2$
7	$\neg \forall Y(Ya \leftrightarrow Yb)$	$\neg I, 3-6$

3. $\exists X(Xa \wedge \neg Xb) \vdash_2 \neg a = b$

1	$\exists X(Xa \wedge \neg Xb)$	
2	$Fa \wedge \neg Fb$	
3	$a = b$	
4	Fa	$\wedge E, 2$
5	$\neg Fb$	$\wedge E, 2$
6	Fb	$=E, 3, 4$
7	\perp	$\perp I, 6, 5$
8	$\neg a = b$	$\neg I, 3-7$
9	$\neg a = b$	$\exists_2 E, 1, 2-8$

4. $\exists X \exists y \neg Xy \vdash_2 \neg \forall X \forall y Xy$

1		$\exists X \exists y \neg Xy$	
2			
3			$\exists y \neg Fy$
4			$\neg \forall y Fy$ CQ, 2
5			$\forall X \forall y Xy$
6			$\forall y Fy$ $\forall_2 E$, 4
7			\perp $\perp I$, 5, 3
8		$\neg \forall X \forall y Xy$	$\neg I$, 4-6
9		$\neg \forall X \forall y Xy$	$\exists_2 E$, 1, 2-7

5. $\neg \exists x \neg x = a \vdash_2 \forall X (Xa \rightarrow \forall x Xx)$

1		$\neg \exists x \neg x = a$	
2			
3			Fa
4			$\forall x \neg \neg x = a$ CQ, 2
5			$b = a$ $\forall_1 E$, 3
6			Fb $=E$, 4, 2
7			$\forall x Fx$ $\forall_1 I$, 5
8		$Fa \rightarrow \forall x Fx$	$\rightarrow I$, 2-6
9		$\forall X (Xa \rightarrow \forall x Xx)$	$\forall_2 I$, 7

6. $\forall X(Xa \rightarrow \forall y(\neg y = a \rightarrow \neg Xy)), Qa \vdash_2 \exists Z \neg \exists x \exists y(\neg x = y \wedge (Zx \wedge Zy))$

1	$\forall X(Xa \rightarrow \forall y(\neg y = a \rightarrow \neg Xy))$	
2	Qa	
3	$Qa \rightarrow \forall y(\neg y = a \rightarrow \neg Qy)$	$\forall_2E, 1$
4	$\forall y(\neg y = a \rightarrow \neg Qy)$	$\rightarrow E, 3, 2$
5	$\exists x \exists y(\neg x = y \wedge (Qx \wedge Qy))$	
6	$\exists y(\neg b = y \wedge (Qb \wedge Qy))$	
7	$\neg b = c \wedge (Qb \wedge Qc)$	
8	$\neg b = c$	$\wedge E, 7$
9	$Qb \wedge Qc$	$\wedge E, 7$
10	$\neg b = a$	
11	$\neg b = a \rightarrow \neg Qb$	$\forall_1E, 4$
12	$\neg Qb$	$\rightarrow E, 11, 10$
13	Qb	$\wedge E, 9$
14	\perp	$\perp I, 13, 12$
15	$\neg \neg b = a$	$\neg I, 10-14$
16	$b = a$	$DNE, 15$
17	$\neg c = a$	
18	$\neg c = a \rightarrow \neg Qc$	$\forall_1E, 4$
19	$\neg Qc$	$\rightarrow E, 18, 17$
20	Qc	$\wedge E, 9$
21	\perp	$\perp I, 20, 19$
22	$\neg \neg c = a$	$\neg I, 17-21$
23	$c = a$	$DNE, 22$
24	$b = c$	$=E, 23, 16$
25	\perp	$\perp I, 24, 8$
26	\perp	$\exists_1E, 6, 7-25$
27	\perp	$\exists_1E, 5, 6-26$
28	$\neg \exists x \exists y(\neg x = y \wedge (Qx \wedge Qy))$	$\neg I, 5-27$
29	$\exists Z \neg \exists x \exists y(\neg x = y \wedge (Zx \wedge Zy))$	$\exists_2I, 28$

Solutions to §3.2

Provide proofs for the following:

1. $Aa, \neg Ab \vdash_2 \exists Y(\neg Ya \wedge Yb)$

1	Aa					
2	$\neg Ab$					
3	$\exists X\forall x(Xx \leftrightarrow \neg Ax)$	Comp				
4	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px; vertical-align: top;">$\forall x(Fx \leftrightarrow \neg Ax)$</td> <td></td> </tr> <tr> <td colspan="2" style="border-top: 1px solid black;"></td> </tr> </table>	$\forall x(Fx \leftrightarrow \neg Ax)$				
$\forall x(Fx \leftrightarrow \neg Ax)$						
5	$Fb \leftrightarrow \neg Ab$	$\forall_1E, 4$				
6	Fb	$\leftrightarrow E, 5, 2$				
7	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px; vertical-align: top;">Fa</td> <td></td> </tr> <tr> <td colspan="2" style="border-top: 1px solid black;"></td> </tr> </table>	Fa				
Fa						
8	$Fa \leftrightarrow \neg Aa$	$\forall_1E, 4$				
9	$\neg Aa$	$\leftrightarrow E, 8, 7$				
10	\perp	$\perp I, 1, 9$				
11	$\neg Fa$	$\neg I, 7-10$				
12	$\neg Fa \wedge Fb$	$\wedge I, 11, 6$				
13	$\exists Y(\neg Ya \wedge Yb)$	$\exists_2I, 12$				
14	$\exists Y(\neg Ya \wedge Yb)$	$\exists_2E, 3, 4-13$				

2. $\vdash_2 \exists X \forall x \neg Xx$

1	$\exists X \forall x (Xx \leftrightarrow \neg x = x)$	Comp						
2	$\forall x (Fx \leftrightarrow \neg x = x)$							
3	$Fa \leftrightarrow \neg a = a$	$\forall_2E, 2$						
4	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">4</td> <td style="width: 10%; border-left: 1px solid black; padding-left: 5px; vertical-align: top;">Fa</td> <td></td> </tr> <tr> <td style="border-top: 1px solid black;"></td> <td style="border-top: 1px solid black;"></td> <td></td> </tr> </table>	4	Fa					
4	Fa							
5	$\neg a = a$	$\leftrightarrow E, 3, 4$						
6	$a = a$	$=I$						
7	\perp	$\perp I, 6, 5$						
8	$\neg Fa$	$\neg I, 4-7$						
9	$\forall x \neg Fx$	$\forall_1I, 8$						
10	$\exists X \forall x \neg Xx$	$\exists_2I, 9$						
11	$\exists X \forall x \neg Xx$	$\exists_2E, 1, 2-10$						

3. $\exists Y \forall X \forall z (Yz \leftrightarrow Xz) \vdash_2 \perp$

1	$\exists Y \forall X \forall z (Yz \leftrightarrow Xz)$	
2	$\forall X \forall z (Fz \leftrightarrow Xz)$	
3	$\exists X \forall x (Xx \leftrightarrow \neg Fx)$	Comp
4	$\forall x (Gx \leftrightarrow \neg Fx)$	
5	$\forall z (Fz \leftrightarrow Gz)$	$\forall_2E, 2$
6	$Fa \leftrightarrow Ga$	$\forall_1E, 5$
7	$Ga \leftrightarrow \neg Fa$	$\forall_1E, 4$
8	Fa	
9	Ga	$\leftrightarrow E, 6, 8$
10	$\neg Fa$	$\leftrightarrow E, 7, 9$
11	\perp	$\perp I, 8, 10$
12	$\neg Fa$	
13	Ga	$\leftrightarrow E, 7, 12$
14	Fa	$\leftrightarrow E, 6, 13$
15	\perp	$\perp I, 14, 12$
16	\perp	TND, 8–11, 12–15
17	\perp	$\exists_2E, 3, 4-16$
18	\perp	$\exists_2E, 1, 2-17$

4. $\vdash_2 \forall Z(\forall Y\forall x(Yx \rightarrow Zx) \rightarrow \forall xZx)$

1		<u>$\forall Y\forall x(Yx \rightarrow Fx)$</u>	
2		<u>$\neg Fa$</u>	
3		$\exists X\forall x(Xx \leftrightarrow x = x)$	Comp
4		<u>$\forall x(Gx \leftrightarrow x = x)$</u>	
5		$Ga \leftrightarrow a = a$	$\forall_1E, 4$
6		$a = a$	$=I$
7		Ga	$\leftrightarrow E, 5, 6$
8		$\forall x(Gx \rightarrow Fx)$	$\forall_2E, 1$
9		$Ga \rightarrow Fa$	$\forall_1E, 8$
10		Fa	$\rightarrow E, 10, 7$
11		\perp	$\perp I, 10, 2$
12		\perp	$\exists_2E, 3, 4-11$
13		$\neg\neg Fa$	$\neg I, 2-12$
14		Fa	DNE, 13
15		$\forall xFx$	$\forall_1I, 14$
16		$\forall Y\forall x(Yx \rightarrow Fx) \rightarrow \forall xFx$	$\rightarrow I, 1-15$
17		$\forall Z(\forall Y\forall x(Yx \rightarrow Zx) \rightarrow \forall xZx)$	$\forall_2I, 16$

5. $\forall X((Xa \wedge Xb) \rightarrow Xc), \forall x\forall y(Rxy \rightarrow Ryx), \exists xRax \wedge \exists xRbx \vdash_2 \exists yRyc$

1	$\forall X((Xa \wedge Xb) \rightarrow Xc)$	
2	$\forall x\forall y(Rxy \rightarrow Ryx)$	
3	$\exists xRax \wedge \exists xRbx$	
4	$\exists xRax$	$\wedge E, 3$
5	$\exists xRbx$	$\wedge E, 3$
6	$\exists X\forall y(Xy \leftrightarrow \exists xRyx)$	Comp
7	$\forall y(Fy \leftrightarrow \exists xRyx)$	
8	$Fa \leftrightarrow \exists xRax$	$\forall_1 E, 7$
9	Fa	$\leftrightarrow E, 8, 4$
10	$Fb \leftrightarrow \exists xRbx$	$\forall_1 E, 7$
11	Fb	$\leftrightarrow E, 10, 5$
12	$Fa \wedge Fb$	$\wedge I, 9, 11$
13	$(Fa \wedge Fb) \rightarrow Fc$	$\forall_2 E, 1$
14	Fc	$\rightarrow E, 13, 12$
15	$Fc \leftrightarrow \exists xRcx$	$\forall_1 E, 7$
16	$\exists xRcx$	$\leftrightarrow E, 15, 14$
17	$\exists xRcx$	$\exists_2 E, 6, 7-16$
18	Rcd	
19	$\forall y(Rcy \rightarrow Ryc)$	$\forall_1 E, 2$
20	$Rcd \rightarrow Rdc$	$\forall_1 E, 19$
21	Rdc	$\rightarrow E, 20, 18$
22	$\exists yRyc$	$\exists_1 I, 21$
23	$\exists yRyc$	$\exists_1 E, 17, 18-22$