Intermediate Logic
Lecture Seven

Natural Deduction for FOL

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Natural Deduction for FOL

Reasoning about *all* Interpretations

- Reasoning about interpretations is great when we want to show that some argument is **not** valid in FOL, or that some sentence is **not** a logical truth
  - All we have to do is come up with a **single** counter-interpretation
- But when we want to show that an argument **is** valid in FOL, or a sentence **is** a logical truth, then they are a lot less helpful
  - To show that $\mathcal{A}$ is a logical truth, we must somehow show that it is true in **all** interpretations

Reasoning about *all* Interpretations

- It is sometimes possible to reason about *all* interpretations, but it is usually **very** hard
- There certainly is not any mechanical method for searching through interpretations (whereas there was a mechanical method for searching through TFL valuations)
- As a result, it is not very practical to use interpretations to show that an argument is valid in FOL, or that a sentence is a logical truth
- Instead, we need to use a different method: *formal proofs!*

Building on TFL Proofs

- This week and next, we will look at how to construct proofs in FOL
- When proving things in FOL, we will use **all** of the rules that we used in TFL
  - That includes basic **and derived** rules!
- All we need to do is add some **extra rules** to the system
  - This week we will add the basic rules for the quantifiers
  - Next week we will add some extra derived rules, plus the basic rules for identity

Introduction and Elimination

- Just like the connectives of TFL, each quantifier is governed by an **Introduction Rule** and an **Elimination Rule**
- Annoyingly, both quantifiers have an easy rule and a hard rule
  - The Introduction Rule for $\exists$ is easy, but the Elimination Rule is hard
  - The Elimination Rule for $\forall$ is easy, but the Introduction Rule is hard
- We will start with the easy rules, and then look at the harder ones later

Natural Deduction for FOL

What can You Infer from a Universal Generalisation?

- Suppose you knew that the following universal generalisation is true:
    - Everyone loves *Intermediate Logic*
- You could then infer that this generalisation holds of each individual person
    - Natasha loves *Intermediate Logic*
    - George loves *Intermediate Logic*
    - Hazel loves *Intermediate Logic*
    - ...
- This leads us to our Universal Elimination Rule

Universal Elimination

$$\begin{array}{r|ll} m & \forall \chi \mathcal{A}(\ldots\chi\ldots\chi\ldots) & \\ & \mathcal{A}(\ldots c\ldots c\ldots) & \forall\text{E}, m \end{array}$$

- $\mathcal{A}(\ldots\chi\ldots\chi\ldots)$ is a formula containing **one or more** occurrences of some variable χ
- c can be any name you like
- $\mathcal{A}(\ldots c\ldots c\ldots)$ is the result of replacing **all** of the occurrences of χ in $\mathcal{A}(\ldots\chi\ldots\chi\ldots)$ with c

Some Examples

8	$\forall x Rax$	
...	...	
15	Rab	∀E, 8

Some Examples

8	$\forall y Ray$	
...	...	
15	Rab	$\forall$E, 8

Some Examples

8	$\forall y Ray$	
…	…	
15	Rac	∀E, 8

Some Examples

8	$\forall y Ray$	
...	...	
15	Raa	∀E, 8

Some Examples

8	$\forall x(Fx \rightarrow (Rax \lor Gb))$	
...	...	
15	$Fa \rightarrow (Raa \lor Gb)$	$\forall$E, 8

Some Examples

8	$\forall x(Fx \to (Rax \lor Gb))$	
...	...	
15	$Fb \to (Rab \lor Gb)$	$\forall$E, 8

Some Examples

8	$\forall x(Fx \rightarrow (Rax \vee Gb))$	
...	...	
15	$Fc \rightarrow (Rac \vee Gb)$	∀E, 8

A *Bad* Example

8	$\forall x(Fx \to (Rax \lor Gb))$	
...	...	
15	$Fc \to (Rax \lor Gb)$	$\forall$E, 8

- This is a *bad* example of Universal Elimination, because we replaced some but **not all** of the ‘*x*’s with ‘*c*’s

Two Universal Eliminations

1	$\forall x \forall y Rxy$	
2	$\forall y Ray$	$\forall$E, 1
3	Rab	$\forall$E, 2

Not One Double Elimination!

1	$\forall x \forall y Rxy$	
2	Rab	∀E, 1

Two Universal Eliminations!!!

1	$\forall x \forall y Rxy$	
2	$\forall y Ray$	∀E, 1
3	Rab	∀E, 2

Natural Deduction for FOL

Introducing Natural Deduction for FOL

Universal Elimimation

Existential Introduction

Universal Introduction

Existential Elimination

How would You Prove an Existential Generalisation?

- Imagine you wanted to prove that an existential generalisation was true, for example:
  - Someone loves *Intermediate Logic*
- A really excellent way of doing this would be by first proving that this generalisation is true of some particular person
  - If you first proved that Noah loves *Intermediate Logic*, you could then infer that *someone* loves *Intermediate Logic*
- This motivates our Existential Introduction rule

Existential Introduction

$$\begin{array}{r|ll} m & \mathcal{A}(\ldots c \ldots c \ldots) & \\ n & \exists \chi \mathcal{A}(\ldots \chi \ldots c \ldots) & \qquad \exists\text{I}, m \end{array}$$

- $\mathcal{A}(\ldots c \ldots c \ldots)$ is a sentence containing **one or more** occurrences of the name c
- χ can be any variable that does **not** occur in $\mathcal{A}(\ldots c \ldots c \ldots)$
- $\mathcal{A}(\ldots \chi \ldots c \ldots)$ is the result of replacing **one or more** of the occurrences of c in $\mathcal{A}(\ldots c \ldots c \ldots)$ with the variable χ

Some Examples

8	$Raba$	
…	…	
15	$\exists x\, Rxba$	∃I, 8

Some Examples

8	$Raba$	
...	...	
15	$\exists x\ Rabx$	$\exists$I, 8

Some Examples

8	$Raba$	
...	...	
15	$\exists x\ Rxbx$	∃I, 8

Some Examples

8	$Raba$	
…	…	
15	$\exists y\, Ryby$	∃I, 8

Some Examples

8 | $Raba$

... | ...

15 | $\exists z\, Rzbz$ ∃I, 8

Some Examples

8	$Pa \to (Fb \lor \neg Sac)$	
...	...	
15	$\exists x(Px \to (Fb \lor \neg Sac))$	∃**I**, 8

Some Examples

8	$Pa \to (Fb \lor \neg Sac)$	
...	...	
15	$\exists x(Pa \to (Fb \lor \neg Sxc))$	$\exists$I, 8

Some Examples

8 | $Pa \rightarrow (Fb \vee \neg Sac)$

… | …

15 | $\exists x(Px \rightarrow (Fb \vee \neg Sxc))$ $\quad$ ∃I, 8

Some Examples

8	$Pa \to (Fb \lor \neg Sac)$	
...	...	
15	$\exists y(Py \to (Fb \lor \neg Syc))$	$\exists$I, 8

Some Examples

8	$Pa \to (Fb \lor \neg Sac)$	
...	...	
15	$\exists z(Pz \to (Fb \lor \neg Szc))$	∃I, 8

A *Bad* Example

$$\begin{array}{l|ll} 8 & Pa \to (Fb \lor \neg Sac) & \\ \ldots & \ldots & \\ 15 & \exists z(Pz \to (Fz \lor \neg Szc)) & \exists\text{I}, 8 \end{array}$$

- This is a *bad* example of Existential Introduction, because we replaced **two different names** (‘*a*’ and ‘*b*’) with the same variable

Two Existential Introductions

1	Rab	
2	$\exists yRay$	$\exists$I, 1
3	$\exists x\exists yRxy$	$\exists$I, 2

Not One Double Introduction!

1	Rab	
2	$\exists x \exists y Rxy$	$\exists$I, 1

Two Existential Introductions!!!

1	Rab	
2	$\exists y Ray$	∃I, 1
3	$\exists x \exists y Rxy$	∃I, 2

Natural Deduction for FOL

How would You Prove a Universal Generalisation?

- Imagine you wanted to prove that a universal generalisation was true, for example:
  - Everyone loves *Intermediate Logic*
- One idea would be to go through everyone in the domain, and prove that the generalisation is true of each of them:
  - Will loves *Intermediate Logic*
  - Emma loves *Intermediate Logic*
  - Joshua loves *Intermediate Logic*
  - ...
- That is a bit longwinded and impractical, but it works well enough in this case, since we are working with a finite domain

How would You Prove a Universal Generalisation?

- But now suppose you wanted to prove a universal generalisations about **infinitely** many things
  - Every number is either odd or even
- You definitely couldn't do **that** by going through all of the numbers one by one!
- But there is another way:
  - Start by letting *a* be an arbitrary number
  - Then prove of *a* that it is either odd or even
  - Then conclude that since *a* was just an arbitrarily chosen number, **every** number must be odd or even
- This leads us to our Universal Introduction Rule

Universal Introduction

$$\begin{array}{r|ll} m & \mathcal{A}(\ldots c \ldots c \ldots) & \\ n & \forall \chi \mathcal{A}(\ldots \chi \ldots \chi \ldots) & \qquad \forall\text{I}, m \end{array}$$

- $\mathcal{A}(\ldots c \ldots c \ldots)$ is a sentence containing one or more occurrences of the name c, and $\mathcal{A}(\ldots \chi \ldots \chi \ldots)$ is the formula that you get when you replace **all** of those occurrences of c with the variable χ

- c **must not** occur in any undischarged assumptions above line n (including the premises of the argument)

- c **must not** occur in $\forall \chi \mathcal{A}(\ldots \chi \ldots \chi \ldots)$

A Good Example

1	Pa	
2	Pa	R, 1
3	$Pa \to Pa$	→I, 1–2
4	$\forall x(Px \to Px)$	∀I, 3

A Bad Example

1	$\forall x\, Rxa$	
2	Raa	∀E, 1
3	$\forall x\, Rxx$	∀I, 2

- This is a bad argument because ‘*a*’ appeared in an undischarged assumption (line 1)
- In this case, we made a background assumption about *a*, and so *a* isn’t really an **arbitrary** object!

Another Bad Example

1	$\forall x\, Rxx$	
2	Raa	$\forall$E, 1
3	$\forall y\, Ray$	$\forall$**I**, 2

- This is a bad argument because we only replaced **some** occurrences of ‘a’ with ‘y’

A Good Example (Again!)

1	$\forall x\, Rxx$	
2	Raa	$\forall$E, 1
3	$\forall y\, Ryy$	$\forall$**I**, 2

- This is a bad argument because we only replaced **some** occurrences of ‘*a*’ with ‘*y*’
- If we replaced **all** of the occurrences of ‘*a*’ with ‘*y*’, the inference would’ve been trivial, but fine

Two Universal Introductions

1	Fa	
2	Fa	R, 1
3	$Fa \to Fa$	$\to$I, 1–2
4	Gb	
5	Gb	R, 4
6	$Gb \to Gb$	$\to$I, 4–5
7	$(Fa \to Fa) \land (Gb \to Gb)$	$\land$I, 3, 6
8	$\forall y((Fa \to Fa) \land (Gy \to Gy))$	$\forall$I, 7
9	$\forall x \forall y((Fx \to Fx) \land (Gy \to Gy))$	$\forall$I, 8

Not One Double Introduction!

$$\begin{array}{r|lr}
1 & \quad \begin{array}{|l}\underline{Fa}\end{array} & \\
2 & \quad \begin{array}{|l}Fa\end{array} & \text{R, 1} \\
3 & Fa \to Fa & \to\text{I, 1–2} \\
4 & \quad \begin{array}{|l}\underline{Gb}\end{array} & \\
5 & \quad \begin{array}{|l}Gb\end{array} & \text{R, 4} \\
6 & Gb \to Gb & \to\text{I, 4–5} \\
7 & (Fa \to Fa) \land (Gb \to Gb) & \land\text{I, 3, 6} \\
8 & \forall x \forall y((Fx \to Fx) \land (Gy \to Gy)) & \forall\text{I, 7}
\end{array}$$

Two Universal Introductions!!!

1	$\quad \mid Fa$	
2	$\quad \mid Fa$	R, 1
3	$Fa \to Fa$	$\to$I, 1–2
4	$\quad \mid Gb$	
5	$\quad \mid Gb$	R, 4
6	$Gb \to Gb$	$\to$I, 4–5
7	$(Fa \to Fa) \wedge (Gb \to Gb)$	$\wedge$I, 3, 6
8	$\forall y((Fa \to Fa) \wedge (Gy \to Gy))$	$\forall$I, 7
9	$\forall x \forall y((Fx \to Fx) \wedge (Gy \to Gy))$	$\forall$I, 8

Natural Deduction for FOL

What can You Infer from an Existential Generalisation?

- Suppose that you knew the following universal generalisation is true:
  - Someone loves *Intermediate Logic*
- You could not infer that that this generalisation holds of any particular person
  - It might be Charlie who loves *Intermediate Logic*, or it might be April, or it might be Kishori...
- So what could you infer?

What can You Infer from an Existential Generalisation?

- You could argue like this:
    - Suppose that April loves *Intermediate Logic*
    - Given that April loves *Intermediate Logic*, she must be attending all the lectures
    - The same would go for anyone else, if they loved *Intermediate Logic*
    - So even if I drop my supposition that April loves *Intermediate Logic*, since I do know that someone loves it, there must be someone who is attending all the lectures
- This motivates our Existential Elimination Rule

Existential Elimination

m	$\exists \chi \mathcal{A}(\ldots\chi\ldots\chi\ldots)$	
n	$\quad \mathcal{A}(\ldots c\ldots c\ldots)$	
o	$\quad \mathcal{B}$	
	$\mathcal{B}$	∃E, m, n–o

- c **must not** occur in any undischarged assumptions above line n (including the premises of the argument)
- c **must not** occur in $\exists \chi \mathcal{A}(\ldots\chi\ldots\chi\ldots)$
- c **must not** appear in $\mathcal{B}$

Existential Elimination

$$\begin{array}{l|l l}
m & \exists \mathcal{X}\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots) & \\
n & \quad \begin{array}{|l} \mathcal{A}(\ldots c\ldots c\ldots) \\ \hline \end{array} & \\
o & \quad \begin{array}{|l} \mathcal{B} \end{array} & \\
 & \mathcal{B} & \exists\text{E}, m, n\text{–}o
\end{array}$$

- c **must not** appear in any line before m
- c **must not** appear in $\mathcal{B}$

An Example

- $\exists x Rax, \forall y(Ray \to Fy) \therefore \exists z Fz$

1	$\exists x Rax$		
2	$\forall y(Ray \to Fy)$		
3		Rab	
4		$Rab \to Fb$	$\forall$E, 2
5		Fb	$\to$E, 4, 3
6		$\exists z Fz$	$\exists$I, 5
7	$\exists z Fz$		$\exists$E, 1, 3–6

A Bad Example!

1	$\exists x Rax$		
2	$\forall y(Ray \to Fy)$		
3		Rab	
4		$Rab \to Fb$	$\forall$E, 2
5		Fb	$\to$E, 4, 3
6	Fb		$\exists$E, 1, 3–5

- This is a bad argument because line 5 contains the name *b*, which is the name we introduced at line 3

Another Bad Example!

1	$\exists x Rax$	
2	$\forall y(Ray \to Fy)$	
3	$\quad Raa$	
4	$\quad Raa \to Fa$	$\forall$E, 2
5	$\quad Fa$	$\to$E, 4, 3
6	$\quad Fa \wedge Raa$	$\wedge$I, 3, 5
7	$\quad \exists x(Fx \wedge Rxx)$	$\exists$I, 6
8	$\exists x(Fx \wedge Rxx)$	$\exists$E, 1, 3–7

- This is a bad proof, because the name we introduced at line 3 already appeared in lines 1 and 2

Two Existential Eliminations

1	$\vert\ \exists x \exists y Rxy$	
2	$\vert\ \underline{\forall x \forall y (Rxy \to Gy)}$	
3	$\vert\ \vert\ \underline{\exists y Ray}$	
4	$\vert\ \vert\ \vert\ \underline{Rab}$	
5	$\vert\ \vert\ \vert\ \forall y (Ray \to Gy)$	∀E, 2
6	$\vert\ \vert\ \vert\ Rab \to Gb$	∀E, 5
7	$\vert\ \vert\ \vert\ Gb$	→E, 6, 4
8	$\vert\ \vert\ \vert\ \exists x Gx$	∃I, 7
9	$\vert\ \vert\ \exists x Gx$	∃E, 3, 4–8
10	$\vert\ \exists x Gx$	∃E, 1, 3–9

Not One Double Elimination!

1	$\exists x \exists y Rxy$	
2	$\forall x \forall y (Rxy \to Gy)$	
3	$\quad Rab$	
4	$\quad \forall y (Ray \to Gy)$	∀E, 2
5	$\quad Rab \to Gb$	∀E, 4
6	$\quad Gb$	→E, 5, 3
7	$\quad \exists x Gx$	∃I, 6
8	$\exists x Gx$	∃E, 1, 3–7

Two Existential Eliminations!!!

1	$\mid \exists x \exists y Rxy$	
2	$\mid \forall x \forall y (Rxy \to Gy)$	
3	$\mid \quad \mid \exists y Ray$	
4	$\mid \quad \mid \quad \mid Rab$	
5	$\mid \quad \mid \quad \mid \forall y (Ray \to Gy)$	∀E, 2
6	$\mid \quad \mid \quad \mid Rab \to Gb$	∀E, 5
7	$\mid \quad \mid \quad \mid Gb$	→E, 6, 4
8	$\mid \quad \mid \quad \mid \exists x Gx$	∃I, 7
9	$\mid \quad \mid \exists x Gx$	∃E, 3, 4–8
10	$\mid \exists x Gx$	∃E, 1, 3–9

Re-Introducing the Single Turnstile

- We will continue to use the single turnstile, '$\vdash$' to express provability
  - It is possible to construct a proof which starts with $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ as premises, and ends with $\mathcal{C}$ as the conclusion
  - $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash \mathcal{C}$
  - It is possible to construct a proof which doesn't have any premises, and ends with $\mathcal{C}$ as the conclusion
  - $\vdash \mathcal{C}$
- But now we can use the rules for quantifiers as well as all the rules you learnt for TFL

Exercises

- Provide a proof for each of the following:

  1. $\vdash \forall z(Pz \vee \neg Pz)$
  2. $\forall x(Ax \rightarrow Bx),\ \exists xAx \ \vdash\ \exists xBx$
  3. $\forall x(Mx \leftrightarrow Nx),\ Ma \wedge \exists xRxa \ \vdash\ \exists xNx$
  4. $\forall x(\neg Mx \vee Ljx),\ \forall x(Bx \rightarrow Ljx),\ \forall x(Mx \vee Bx) \ \vdash\ \forall xLjx$
  5. $\forall x\forall yGxy \ \vdash\ \exists xGxx$