Intermediate Logic
Lecture Three

More Natural Deduction for TFL

Rob Trueman
rob.trueman@york.ac.uk

University of York

More Natural Deduction for TFL

The Very Idea of a Formal Proof

- Last week we started looking at how to construct formal proofs in TFL
- You can think of a building a formal proof as a kind of game:
  - You start with a collection of premises
  - You aim to get from these premises to the conclusion
  - But every move you make has to be allowed by a set of rules
- (Nearly) all the rules come in one of two kinds:
  - **Introduction Rules** allow you to introduce a connective into a sentence
  - **Elimination Rules** allow you to eliminate a connective from a sentence

An Example

$$A \vee B,\ \neg A,\ B \to C\ \therefore\ C$$

1	$A \vee B$	
2	$\neg A$	
3	$B \to C$	
4	$\quad A$	
5	$\quad \bot$	$\bot$I, 4, 2
6	$\quad C$	$\bot$E, 5
7	$\quad B$	
8	$\quad C$	$\to$E, 3, 7
9	C	$\vee$E, 1, 4–6, 7–8

Proof Strategies

(1) Figure out what the main connective in your conclusion is; one plan is to think about how you would introduce that connective

(2) Look at what you already have; it may be that you can make progress by applying some elimination rules

(3) Don't be afraid to try making new assumptions

(4) If all else fails, try using Tertium Non Datur; some proofs require you to use that rule

(5) If even that fails, then there is nothing for it but to **JUST KEEP TRYING!!!**

Exercises

Give a proof for each of the following arguments

1. $P \wedge (Q \vee R), P \rightarrow \neg R \therefore Q \vee E$
2. $\neg(P \rightarrow Q) \therefore \neg Q$
3. $\neg(P \rightarrow Q) \therefore P$

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The Rules are too Restrictive!

- The rules we have been using so far are annoyingly restrictive and fiddly
- It is just obvious that $\mathcal{A}$ implies $\mathcal{A}$, but to prove it, we have to go round the houses:

1	$\mathcal{A}$	
2	$\mathcal{A} \wedge \mathcal{A}$	$\wedge$I, 1, 1
3	$\mathcal{A}$	$\wedge$E, 2

- This is obviously far too pedantic, and so we will add some extra rules to make our formal system much easier to use

Reiteration

m	$\mathcal{A}$	
	$\mathcal{A}$	R, m

- This rule might seem absolutely trivial and pointless, but as we saw when we were doing our exercises, having that rule does speed proofs up!

Disjunctive Syllogism

- Here is an obviously valid argument:
  - Sharon either studies archaeology or she has a million pounds
  - Sharon does not have a million pounds
  - Therefore, Sharon studies archaeology
- This pattern of inference is known as **Disjunctive Syllogism**
  - $\mathcal{A} \vee \mathcal{B}$, $\neg\mathcal{B}$ $\therefore$ $\mathcal{A}$
- If we know that either $\mathcal{A}$ is true or $\mathcal{B}$ is true, and we also know that $\mathcal{B}$ isn't true, then we know that $\mathcal{A}$ must be true!

Disjunctive Syllogism

m	$\mathcal{A} \vee \mathcal{B}$	
n	$\neg\mathcal{A}$	
	$\mathcal{B}$	DS, m, n

m	$\mathcal{A} \vee \mathcal{B}$	
n	$\neg\mathcal{B}$	
	$\mathcal{A}$	DS, m, n

Modus Tollens

- Here is an obviously valid argument:
  - If Sharon studies archaeology, then she tells Rob a lot about old pots
  - Sharon does not tell Rob a lot about old pots
  - Therefore, Sharon does not study archaeology
- This pattern of inference is known as **Modus Tollens**
  - $\mathcal{A} \rightarrow \mathcal{B}, \neg\mathcal{B} \therefore \neg\mathcal{A}$

Modus Tollens

m	$\mathcal{A} \rightarrow \mathcal{B}$	
n	$\neg\mathcal{B}$	
	$\neg\mathcal{A}$	MT, m, n

Not to be Confused with...

- It is really important not to confuse Modus Tollens (which is a valid argument form) with the following (which is an invalid argument form):
  - $\mathcal{A} \to \mathcal{B}, \neg\mathcal{A} \therefore \neg\mathcal{B}$
- Here is an example of this bad reasoning:
  - If it is raining outside, then Simon is miserable
  - It is not raining outside
  - Therefore, Simon is not miserable
- This is not a valid argument: something *else* might have made Simon miserable!

Double-Negation Elimination

m	$\neg\neg\mathcal{A}$	
	$\mathcal{A}$	DNE, m

- Interestingly, if we wanted to, we could have used DNE as a basic rule instead of TND, and the resulting system would've been exactly the same
- Some logicians, called **intuitionists**, reject DNE and TND

The De Morgan Rules

- The last rules to add are known as De Morgan's Laws, named after Augustus De Morgan, a 19th Century British logician and mathematician
- These rules all govern the way that negation interacts with conjunction and disjunction
- Here is one example:
  - It is not the case that (grass is white **or** snow is green)
  - Therefore, grass is not white **and** snow is not green
- Here is another:
  - It is not the case that Sharon **both** studies archaeology **and** has a million pounds
  - Therefore, **either** Sharon does not study archaeology, **or** Sharon does not have a million pounds

The De Morgan Rules

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

m	$\neg\mathcal{A} \vee \neg\mathcal{B}$	
	$\neg(\mathcal{A} \wedge \mathcal{B})$	DeM, m

m	$\neg(\mathcal{A} \vee \mathcal{B})$	
	$\neg\mathcal{A} \wedge \neg\mathcal{B}$	DeM, m

m	$\neg\mathcal{A} \wedge \neg\mathcal{B}$	
	$\neg(\mathcal{A} \vee \mathcal{B})$	DeM, m

Exercises!

Give a proof for each of these arguments:

1. $E \vee F$, $F \vee G$, $\neg F$ $\therefore$ $E \wedge G$
2. $M \vee (N \rightarrow M)$ $\therefore$ $\neg M \rightarrow \neg N$
3. $(M \vee N) \wedge (O \vee P)$, $N \rightarrow P$, $\neg P$ $\therefore$ $M \wedge O$
4. $(X \wedge Y) \vee (X \wedge Z)$, $\neg(X \wedge D)$, $D \vee M$ $\therefore$ M

More Natural Deduction for TFL

The Additional Rules are Just Shortcuts

- Why are we free to add all of these extra rules to our proof system?
- These additional rules do not add any power to the proof system
  - If you can prove something using the additional rules, you could prove it just using the basic rules too
- The additional rules are short cuts, which just let us prove things a little more quickly
- We can prove this by showing how we can **derive** the additional rules from the basic rules

Deriving Reiteration

m	$\mathcal{A}$	
	$\mathcal{A}$	R, m

1	$\mathcal{A}$	
2	$\mathcal{A} \wedge \mathcal{A}$	$\wedge$I, 1, 1
3	$\mathcal{A}$	$\wedge$E, 2

Deriving Disjunctive Syllogism

m	$\mathcal{A} \vee \mathcal{B}$	
n	$\neg\mathcal{A}$	
	$\mathcal{B}$	DS, m, n

1	$\mathcal{A} \vee \mathcal{B}$	
2	$\neg\mathcal{A}$	
3	$\quad \mathcal{A}$	
4	$\quad \bot$	$\bot$I, 3, 2
5	$\quad \mathcal{B}$	$\bot$E, 4
6	$\quad \mathcal{B}$	
7	$\quad \mathcal{B}$	R, 6
8	$\mathcal{B}$	$\vee$E, 1, 3–5, 6–7

Deriving Modus Tollens

- Now we will derive Modus Tollens together

m	$\mathcal{A} \to \mathcal{B}$	
n	$\neg\mathcal{B}$	
	$\neg\mathcal{A}$	MT, m, n

Deriving Double-Negation Elimination

- Now you can derive Double-Negation Elimination in pairs

m	$\neg\neg\mathcal{A}$	
	$\mathcal{A}$	DNE, m

Deriving the First De Morgan Rule

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

1	$\neg(\mathcal{A} \wedge \mathcal{B})$	
2	$\quad \mathcal{A}$	
3	$\quad\quad \mathcal{B}$	
4	$\quad\quad \mathcal{A} \wedge \mathcal{B}$	$\wedge$I, 2, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\mathcal{B}$	$\neg$I, 3–5
7	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 6
8	$\quad \neg\mathcal{A}$	

Deriving the First De Morgan Rule

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

1	$\neg(\mathcal{A} \wedge \mathcal{B})$	
2	$\quad \mathcal{A}$	
3	$\quad\quad \mathcal{B}$	
4	$\quad\quad \mathcal{A} \wedge \mathcal{B}$	$\wedge$I, 2, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\mathcal{B}$	$\neg$I, 3–5
7	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 6
8	$\quad \neg\mathcal{A}$	

Deriving the First De Morgan Rule

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

1	$\neg(\mathcal{A} \wedge \mathcal{B})$	
2	$\quad \mathcal{A}$	
3	$\quad\quad \mathcal{B}$	
4	$\quad\quad \mathcal{A} \wedge \mathcal{B}$	$\wedge$I, 2, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\mathcal{B}$	$\neg$I, 3–5
7	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 6
8	$\quad \neg\mathcal{A}$	

Deriving the First De Morgan Rule

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

1	$\neg(\mathcal{A} \wedge \mathcal{B})$	
2	$\quad \mathcal{A}$	
3	$\quad\quad \mathcal{B}$	
4	$\quad\quad \mathcal{A} \wedge \mathcal{B}$	$\wedge$I, 2, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\mathcal{B}$	$\neg$I, 3–5
7	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 6
8	$\quad \neg\mathcal{A}$	

Deriving the First De Morgan Rule

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

1	$\neg(\mathcal{A} \wedge \mathcal{B})$	
2	$\quad \mathcal{A}$	
3	$\quad\quad \mathcal{B}$	
4	$\quad\quad \mathcal{A} \wedge \mathcal{B}$	$\wedge$I, 2, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\mathcal{B}$	$\neg$I, 3–5
7	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 6
8	$\quad \neg\mathcal{A}$	

Deriving the First De Morgan Rule

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

1	$\neg(\mathcal{A} \wedge \mathcal{B})$	
2	$\quad \mathcal{A}$	
3	$\quad\quad \mathcal{B}$	
4	$\quad\quad \mathcal{A} \wedge \mathcal{B}$	$\wedge$I, 2, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\mathcal{B}$	$\neg$I, 3–5
7	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 6
8	$\quad \neg\mathcal{A}$	

Deriving the First De Morgan Rule

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

1	$\neg(\mathcal{A} \wedge \mathcal{B})$	
2	$\quad \mathcal{A}$	
3	$\quad\quad \mathcal{B}$	
4	$\quad\quad \mathcal{A} \wedge \mathcal{B}$	$\wedge$I, 2, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\mathcal{B}$	$\neg$I, 3–5
7	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 6
8	$\quad \neg\mathcal{A}$	

Deriving the First De Morgan Rule

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

1	$\neg(\mathcal{A} \wedge \mathcal{B})$	
2	$\quad \mathcal{A}$	
3	$\quad\quad \mathcal{B}$	
4	$\quad\quad \mathcal{A} \wedge \mathcal{B}$	$\wedge$I, 2, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\mathcal{B}$	$\neg$I, 3–5
7	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 6
8	$\quad \neg\mathcal{A}$	

Deriving the First De Morgan Rule

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

1	$\neg(\mathcal{A} \wedge \mathcal{B})$	
2	$\quad \mathcal{A}$	
3	$\quad\quad \mathcal{B}$	
4	$\quad\quad \mathcal{A} \wedge \mathcal{B}$	$\wedge$I, 2, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\mathcal{B}$	$\neg$I, 3–5
7	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 6
8	$\quad \neg\mathcal{A}$	

Deriving the First De Morgan Rule

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

1	$\neg(\mathcal{A} \wedge \mathcal{B})$	
2	$\quad \mathcal{A}$	
3	$\quad\quad \mathcal{B}$	
4	$\quad\quad \mathcal{A} \wedge \mathcal{B}$	$\wedge$I, 2, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\mathcal{B}$	$\neg$I, 3–5
7	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 6
8	$\quad \neg\mathcal{A}$	
9	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 8

Deriving the First De Morgan Rule

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

1	$\neg(\mathcal{A} \wedge \mathcal{B})$	
2	$\quad \mathcal{A}$	
3	$\quad\quad \mathcal{B}$	
4	$\quad\quad \mathcal{A} \wedge \mathcal{B}$	$\wedge$I, 2, 3
5	$\quad\quad \bot$	$\bot$I, 1, 4
6	$\quad \neg\mathcal{B}$	$\neg$I, 3–5
7	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 6
8	$\quad \neg\mathcal{A}$	
9	$\quad \neg\mathcal{A} \vee \neg\mathcal{B}$	$\vee$I, 8
10	$\neg\mathcal{A} \vee \neg\mathcal{B}$	TND, 2–7, 8–9

Deriving the Second De Morgan Rule

- Now we will derive the second De Morgan Rule together

m	$\neg\mathcal{A} \vee \neg\mathcal{B}$	
	$\neg(\mathcal{A} \wedge \mathcal{B})$	DeM, m

Deriving the Remaining De Morgan Rules

- Now everyone can try to derive one of the remaining De Morgan rules in pairs

m	$\neg(\mathcal{A} \vee \mathcal{B})$	
	$\neg\mathcal{A} \wedge \neg\mathcal{B}$	DeM, m

m	$\neg\mathcal{A} \wedge \neg\mathcal{B}$	
	$\neg(\mathcal{A} \vee \mathcal{B})$	DeM, m

More Natural Deduction for TFL

The Single-Turnstile, $\vdash$

- We will use '$\vdash$' to express provability
  - We can formally prove $\mathcal{C}$ from $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$
  - $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash \mathcal{C}$
- Sometimes we can prove $\mathcal{A}$ without using any premises at all
  - In that case, we say that $\mathcal{A}$ is a theorem
  - Using the single turnstile: $\vdash \mathcal{A}$

Proving a Theorem

$\vdash (Q \to \neg Q) \to \neg Q$

1	$\quad \mid Q \to \neg Q$	
2	$\quad \mid \quad \mid Q$	
3	$\quad \mid \quad \mid \neg Q$	$\to$E, 1, 2
4	$\quad \mid \quad \mid \bot$	$\bot$I, 2, 3
5	$\quad \mid \neg Q$	$\neg$I, 2–4
6	$(Q \to \neg Q) \to \neg Q$	$\to$I, 1–5

$\vdash$ versus $\rightarrow$

- Importantly, '$\vdash$' is **not** a new addition to the object-language TFL
- '$\vdash$' is an addition to the meta-language we are using to talk about TFL
- It is especially important not to confuse '$\vdash$' with '$\rightarrow$'
  - '$\rightarrow$' belongs to the object-language, TFL, and expresses the material conditional
  - '$\vdash$' belongs to the metalanguage, and expresses provability
- Nonetheless, there is an important connection between '$\vdash$' and '$\rightarrow$':
  - $\mathcal{A} \vdash \mathcal{B}$ iff $\vdash \mathcal{A} \rightarrow \mathcal{B}$

$\vdash$ versus $\vDash$

- It is also **vitally** important not to confuse '$\vdash$' with '$\vDash$'
  - '$\vdash$' expresses provability, and is all about constructing formal proofs according to the rules we have laid out
  - '$\vDash$' expresses tautological entailment, and is all about truth tables and valuations
- Of course, we want there to be some link between '$\vdash$' and '$\vDash$'
- After all, we want to be able to use our formal proofs to test for tautological entailment!

Soundness and Completeness

- **Soundness:**
  - If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash \mathcal{C}$
- **Completeness:**
  - If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash \mathcal{C}$

- It turns out that our proof system is sound and complete
- As a result, we can move back and forth between claims about provability and claims about tautological entailment

The Difference Still Matters!

- But that doesn't mean that the difference between '$\vdash$' and '$\vDash$' isn't important
- '$\vdash$' and '$\vDash$' still **mean** completely different things
- Soundness and completeness results aren't just given, they have to be **proved**, and that is not entirely easy
- What is more, there are some formal systems which are not both sound and complete!

A Couple More Concepts

- $\mathcal{A}$ and $\mathcal{B}$ are **provably equivalent** iff $\mathcal{A} \vdash \mathcal{B}$ and $\mathcal{B} \vdash \mathcal{A}$
- $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are **jointly contrary** iff $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash \bot$

Limits of Proofs

- Proofs are great for showing that some conclusion is provable from some premises, or that a pair of sentences are provably equivalent, or that a collection of sentences are jointly contrary
- But it is a lot harder to show that some conclusion is **not** provable from some premises, or that a pair of sentences are **not** provably equivalent, or that a collection of sentences are **not** jointly contrary
- To show that a conclusion is not provable from some sentences, you would need to find some way of showing that there is **no possible** proof from the premises to the conclusion
- **Question for you: Is there a clever way of using soundness and truth tables to do that?**

Exercises!!!

Present proofs to show each of the following:

1. $\vdash O \rightarrow O$
2. $\vdash N \vee \neg N$
3. $\vdash J \leftrightarrow [J \vee (L \wedge \neg L)]$
4. $\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$