forallχ

University of York
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Solutions Booklet

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This booklet contains model answers to the practice exercises found in forallχ:York. For several of the questions, there are multiple correct possible answers; in each case, this booklet contains just one answer. Answers are given in blue; please contact Rob Trueman at rob.trueman@york.ac.uk if you have accessibility requirements.

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Contents

Arguments 1

Highlight the phrase which expresses the conclusion of each of these arguments:

1. It is sunny. So I should take my sunglasses.
2. It must have been sunny. I did wear my sunglasses, after all.
3. No one but you has had their hands in the cookie-jar. And the scene of the crime is littered with cookie-crumbs. You're the culprit!
4. Miss Scarlett and Professor Plum were in the study at the time of the murder. And Reverend Green had the candlestick in the ballroom, and we know that there is no blood on his hands. Hence Colonel Mustard did it in the kitchen with the lead-piping. Recall, after all, that the gun had not been fired.

Valid arguments 2

A. Which of the following arguments is valid? Which is invalid?

1. Hypatia is a mathematician.
2. All mathematicians are carrots.

So: Therefore, Hypatia is a carrot. Valid

1. Abe Lincoln was either 5ft tall or he was once president.
2. Abe Lincoln was never president.

So: Abe Lincoln was5ft tall. Valid

1. If Ingrid trained hard, then she will win the race.
2. Ingrid did not train hard.

So: Ingrid will not win the race. Invalid
Imagine that Ingrid is racing against people *much* slower than her; in that case, she would win even if she didn't train hard.

1. Hugh Jackman was born in either France or Luxemborg.
2. Hugh Jackman was not born in Luxemborg.

So: Hugh Jackman was born in France. Valid

1. If the world were to end today, then I would not need to get up tomorrow morning.
2. I will need to get up tomorrow morning.

So: The world will not end today. Valid

B. Could there be:

1. A valid argument that has one false premise and one true premise? Yes.
   Example: the first argument, above.
2. A valid argument that has only false premises? Yes.
   Example: Socrates is a frog, all frogs are excellent pianists, therefore Socrates is an excellent pianist.
3. A valid argument with only false premises and a false conclusion? Yes.
   The same example will suffice.
4. A sound argument with a false conclusion? No.
   By definition, a sound argument has true premises. And a valid argument is one where it is impossible for the premises to be true and the conclusion false. So the conclusion of a sound argument is certainly true.

5. An invalid argument that can be made valid by the addition of a new premise? Yes.
Plenty of examples, but let me offer a more general observation. We can *always* make an invalid argument valid, by adding a contradiction into the premises. For an argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. If the premises are contradictory, then it is impossible for them all to be true (and the conclusion false).
6. A valid argument that can be made invalid by the addition of a new premise? No.
An argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. Adding another premise will only make it harder for the premises all to be true together.

In each case: if so, give an example; if not, explain why not.

Other logical notions 3

A. For each of the following: Is it necessarily true, necessarily false, or contingent?

1. Caesar crossed the Rubicon. Contingent
2. Someone once crossed the Rubicon. Contingent
3. No one has ever crossed the Rubicon. Contingent
4. If Caesar crossed the Rubicon, then someone has. Necessarily true
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon. Necessarily false
6. If anyone has ever crossed the Rubicon, it was Caesar. Contingent

B. Look back at the sentences G1–G4 in this section (about giraffes, gorillas and martians in the wild animal park), and consider each of the following:

1. G2, G3, and G4 Jointly consistent
2. G1, G3, and G4 Jointly inconsistent
3. G1, G2, and G4 Jointly consistent
4. G1, G2, and G3 Jointly consistent

Which are jointly consistent? Which are jointly inconsistent?

C. Could there be:

1. A valid argument, the conclusion of which is necessarily false?
   Yes: '$1 + 1 = 3$. So $1 + 2 = 4$.'
2. An invalid argument, the conclusion of which is necessarily true?
   No. If the conclusion is necessarily true, then there is no way to make it false, and hence no way to make it false whilst making all the premises true.
3. Jointly consistent sentences, one of which is necessarily false?
   No. If a sentence is necessarily false, there is no way to make it true, let alone along with all the other sentences.
4. Jointly inconsistent sentences, one of which is necessarily true?
   Yes. '$1 + 1 = 4$' and '$1 + 1 = 2$'.

In each case: if so, give an example; if not, explain why not.

Connectives 5

A. Using the symbolisation key given, symbolise each English sentence in TFL.

M: Those creatures are men in suits.
C: Those creatures are chimpanzees.
G: Those creatures are gorillas.

1. Those creatures are not men in suits.
   $\neg M$
2. Those creatures are men in suits, or they are not.
   $(M \vee \neg M)$
3. Those creatures are either gorillas or chimpanzees.
   $(G \vee C)$
4. Those creatures are neither gorillas nor chimpanzees.
   $\neg(C \vee G)$
5. If those creatures are chimpanzees, then they are neither gorillas nor men in suits.
   $(C \rightarrow \neg(G \vee M))$
6. Unless those creatures are men in suits, they are either chimpanzees or they are gorillas.
   $(M \vee (C \vee G))$

B. Using the symbolisation key given, symbolise each English sentence in TFL.

A: Mister Ace was murdered.
B: The butler did it.
C: The cook did it.
D: The Duchess is lying.
E: Mister Edge was murdered.
F: The murder weapon was a frying pan.

1. Either Mister Ace or Mister Edge was murdered.
   $(A \vee E)$
2. If Mister Ace was murdered, then the cook did it.
   $(A \rightarrow C)$
3. If Mister Edge was murdered, then the cook did not do it.
   $(E \rightarrow \neg C)$
4. Either the butler did it, or the Duchess is lying.
   $(B \vee D)$
5. The cook did it only if the Duchess is lying.

$(C \to D)$

6. If the murder weapon was a frying pan, then the culprit must have been the cook.
   $(F \to C)$
7. If the murder weapon was not a frying pan, then the culprit was either the cook or the butler.
   $(\neg F \to (C \lor B))$
8. Mister Ace was murdered if and only if Mister Edge was not murdered.
   $(A \leftrightarrow \neg E)$
9. The Duchess is lying, unless it was Mister Edge who was murdered.
   $(D \lor E)$
10. If Mister Ace was murdered, he was done in with a frying pan.
    $(A \to F)$
11. Since the cook did it, the butler did not.
    $(C \land \neg B)$
12. Of course the Duchess is lying!
    D

C. Using the symbolisation key given, symbolise each English sentence in TFL.

E_1: Ava is an electrician.
E_2: Harrison is an electrician.
F_1: Ava is a firefighter.
F_2: Harrison is a firefighter.
S_1: Ava is satisfied with her career.
S_2: Harrison is satisfied with his career.

1. Ava and Harrison are both electricians.
   $(E_1 \land E_2)$
2. If Ava is a firefighter, then she is satisfied with her career.
   $(F_1 \to S_1)$
3. Ava is a firefighter, unless she is an electrician.
   $(F_1 \lor E_1)$
4. Harrison is an unsatisfied electrician.
   $(E_2 \land \neg S_2)$
5. Neither Ava nor Harrison is an electrician.
   $\neg(E_1 \lor E_2)$
6. Both Ava and Harrison are electricians, but neither of them find it satisfying.
   $((E_1 \land E_2) \land \neg(S_1 \lor S_2))$
7. Harrison is satisfied only if he is a firefighter.
   $(S_2 \to F_2)$
8. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.
   $((\neg E_1 \to \neg E_2) \land (E_1 \to E_2))$
9. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
   $(S_1 \leftrightarrow \neg S_2)$
10. If Harrison is both an electrician and a firefighter, then he must be satisfied with his work.

$((E_2 \wedge F_2) \to S_2)$

11. It cannot be that Harrison is both an electrician and a firefighter.
$\neg(E_2 \wedge F_2)$
12. Harrison and Ava are both firefighters if and only if neither of them is an electrician.
$((F_2 \wedge F_1) \leftrightarrow \neg(E_2 \vee E_1))$

D. Give a symbolisation key and symbolise the following English sentences in TFL.

A: Alice is a spy.
B: Bob is a spy.
C: The code has been broken.
G: The German embassy will be in an uproar.

1. Alice and Bob are both spies.
$(A \wedge B)$
2. If either Alice or Bob is a spy, then the code has been broken.
$((A \vee B) \to C)$
3. If neither Alice nor Bob is a spy, then the code remains unbroken.
$(\neg(A \vee B) \to \neg C)$
4. The German embassy will be in an uproar, unless someone has broken the code.
$(G \vee C)$
5. Either the code has been broken or it has not, but the German embassy will be in an uproar regardless.
$((C \vee \neg C) \wedge G)$
6. Either Alice or Bob is a spy, but not both.
$((A \vee B) \wedge \neg(A \wedge B))$

E. Give a symbolisation key and symbolise the following English sentences in TFL.

F: There is food to be found in the pridelands.
R: Rafiki will talk about squashed bananas.
A: Simba is alive.
K: Scar will remain as king.

1. If there is food to be found in the pridelands, then Rafiki will talk about squashed bananas.
$(F \to R)$
2. Rafiki will talk about squashed bananas unless Simba is alive.
$(R \vee A)$
3. Rafiki will either talk about squashed bananas or he won't, but there is food to be found in the pridelands regardless.
$((R \vee \neg R) \wedge F)$
4. Scar will remain as king if and only if there is food to be found in the pridelands.

$(K \leftrightarrow F)$

5. If Simba is alive, then Scar will not remain as king.
$(A \rightarrow \neg K)$

F. For each argument, write a symbolisation key and symbolise all of the sentences of the argument in TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cranky. Dorothy plays piano in the morning unless she is distracted. So if Roger does not wake up cranky, then Dorothy must be distracted.

   P: Dorothy plays the Piano in the morning.
   C: Roger wakes up cranky.
   D: Dorothy is distracted.

   $(P \rightarrow C)$, $(P \vee D)$, $(\neg C \rightarrow D)$
2. It will either rain or snow on Tuesday. If it rains, Neville will be sad. If it snows, Neville will be cold. Therefore, Neville will either be sad or cold on Tuesday.

   T_1: It rains on Tuesday
   T_2: It snows on Tuesday
   S: Neville is sad on Tuesday
   C: Neville is cold on Tuesday

   $(T_1 \vee T_2)$, $(T_1 \rightarrow S)$, $(T_2 \rightarrow C)$, $(S \vee C)$
3. If Zoog remembered to do his chores, then things are clean but not neat. If he forgot, then things are neat but not clean. Therefore, things are either neat or clean; but not both.

   Z: Zoog remembered to do his chores
   C: Things are clean
   N: Things are neat

   $(Z \rightarrow (C \wedge \neg N))$, $(\neg Z \rightarrow (N \wedge \neg C))$, $((N \vee C) \wedge \neg(N \wedge C))$.

G. We symbolised an *exclusive or* using '$\vee$', '$\wedge$', and '$\neg$'. How could you symbolise an *exclusive or* using only two connectives? Is there any way to symbolise an *exclusive or* using only one connective?
For two connectives, we could offer any of the following:

$$\neg(\mathcal{A} \leftrightarrow \mathcal{B})$$
$$(\neg\mathcal{A} \leftrightarrow \mathcal{B})$$
$$(\neg(\neg\mathcal{A} \wedge \neg\mathcal{B}) \wedge \neg(\mathcal{A} \wedge \mathcal{B}))$$

But if we wanted to symbolise it using only one connective, we would have to introduce a new primitive connective.

Sentences of TFL — 6

A. For each of the following: (a) Is it a sentence of TFL, strictly speaking? (b) Is it a sentence of TFL, allowing for our relaxed bracketing conventions?

1. (A) — (a) no (b) no
2. $J_{374} \vee \neg J_{374}$ — (a) no (b) yes
3. $\neg\neg\neg\neg F$ — (a) yes (b) yes
4. $\neg \wedge S$ — (a) no (b) no
5. $(G \wedge \neg G)$ — (a) yes (b) yes
6. $(A \rightarrow (A \wedge \neg F)) \vee (D \leftrightarrow E)$ — (a) no (b) yes
7. $[(Z \leftrightarrow S) \rightarrow W] \wedge [J \vee X]$ — (a) no (b) yes
8. $(F \leftrightarrow \neg D \rightarrow J) \vee (C \wedge D)$ — (a) no (b) no

B. Are there any sentences of TFL that contain no atomic sentences? Explain your answer.

No. Atomic sentences contain atomic sentences (trivially). And every more complicated sentence is built up out of less complicated sentences, that were in turn built out of less complicated sentences, ..., that were ultimately built out of atomic sentences.

C. What is the scope of each connective in the sentence

$$\big[(H \rightarrow I) \vee (I \rightarrow H)\big] \wedge (J \vee K)$$

The scope of the left-most instance of '$\rightarrow$' is '$(H \rightarrow I)$'.
The scope of the right-most instance of '$\rightarrow$' is '$(I \rightarrow H)$'.
The scope of the left-most instance of '$\vee$ is '$\big[(H \rightarrow I) \vee (I \rightarrow H)\big]$'
The scope of the right-most instance of '$\vee$' is '$(J \vee K)$'
The scope of the conjunction is the entire sentence; so conjunction is the main logical connective of the sentence.

Truth-tables for compound sentences 10

A. Present truth-tables for each of the following:

1. $A \to A$

A	A	$\to$	A
T	T	**T**	T
F	F	**T**	F

2. $C \to \neg C$

C	C	$\to$	$\neg$	C
T	T	**F**	F	T
F	F	**T**	T	F

3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$

A	B	$(A$	$\leftrightarrow$	$B)$	$\leftrightarrow$	$\neg$	$(A$	$\leftrightarrow$	$\neg$	$B)$
T	T	T	T	T	**T**	T	T	F	F	T
T	F	T	F	F	**T**	F	T	T	T	F
F	T	F	F	T	**T**	F	F	T	F	T
F	F	F	T	F	**T**	T	F	F	T	F

4. $(A \to B) \lor (B \to A)$

A	B	$(A$	$\to$	$B)$	$\lor$	$(B$	$\to$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	T	T	**T**	T	F	F
F	F	F	T	F	**T**	F	T	F

5. $(A \land B) \to (B \lor A)$

A	B	$(A$	$\land$	$B)$	$\to$	$(B$	$\lor$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	F	T	**T**	T	T	F
F	F	F	F	F	**T**	F	F	F

6. $\neg(A \lor B) \leftrightarrow (\neg A \land \neg B)$

A	B	$\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
T	T	F T T T **T** F T F F T
T	F	F T T F **T** F T F T F
F	T	F F T T **T** T F F F T
F	F	T F F F **T** T F T T F

7. $\big[(A \wedge B) \wedge \neg(A \wedge B)\big] \wedge C$

A	B	C	$\big[(A \wedge B) \wedge \neg(A \wedge B)\big] \wedge C$
T	T	T	T T T F F T T T **F** T
T	T	F	T T T F F T T T **F** F
T	F	T	T F F F T T F F **F** T
T	F	F	T F F F T T F F **F** F
F	T	T	F F T F T F F F **F** T
F	T	F	F F T F T F F T **F** F
F	F	T	F F F F T F F T **F** T
F	F	F	F F F F T F F F **F** F

8. $[(A \wedge B) \wedge C] \rightarrow B$

A	B	C	$[(A \wedge B) \wedge C] \rightarrow B$
T	T	T	T T T T T **T** T
T	T	F	T T T F F **T** T
T	F	T	T F F F T **T** F
T	F	F	T F F F F **T** F
F	T	T	F F T F T **T** T
F	T	F	F F T F F **T** T
F	F	T	F F F F T **T** F
F	F	F	F F F F F **T** F

9. $\neg\big[(C \vee A) \vee B\big]$

A	B	C	$\neg\big[(C \vee A) \vee B\big]$
T	T	T	**F** T T T T T
T	T	F	**F** F T T T T
T	F	T	**F** T T T T F
T	F	F	**F** F T T T F
F	T	T	**F** T T F T T
F	T	F	**F** F F F T T
F	F	T	**F** T T F T F
F	F	F	**T** F F F F F

B. Check all the claims made in introducing the new notational conventions in §10.3, i.e. show that:

1. '$((A \wedge B) \wedge C)$' and '$(A \wedge (B \wedge C))$' have the same truth-table

A	B	C	$(A \wedge B) \wedge C$	$A \wedge (B \wedge C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **F** T F F
T	F	T	T F F **F** T	T **F** F F T
T	F	F	T F F **F** F	T **F** F F F
F	T	T	F F T **F** T	F **F** T T T
F	T	F	F F T **F** F	F **F** T F F
F	F	T	F F F **F** T	F **F** F F T
F	F	F	F F F **F** F	F **F** F F F

2. '$((A \vee B) \vee C)$' and '$(A \vee (B \vee C))$' have the same truth-table

A	B	C	$(A \vee B) \vee C$	$A \vee (B \vee C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **T** F	T **T** T T F
T	F	T	T T F **T** T	T **T** F T T
T	F	F	T T F **T** F	T **T** F F F
F	T	T	F T T **T** T	F **T** T T T
F	T	F	F T T **T** F	F **T** T T F
F	F	T	F F F **T** T	F **T** F T T
F	F	F	F F F **F** F	F **F** F F F

3. '$((A \vee B) \wedge C)$' and '$(A \vee (B \wedge C))$' do not have the same truth-table

A	B	C	$(A \vee B) \wedge C$	$A \vee (B \wedge C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **T** T F F
T	F	T	T T F **T** T	T **T** F F T
T	F	F	T T F **F** F	T **T** F F F
F	T	T	F T T **T** T	F **T** T T T
F	T	F	F T T **F** F	F **F** T F F
F	F	T	F F F **F** T	F **F** F F T
F	F	F	F F F **F** F	F **F** F F F

4. '$((A \rightarrow B) \rightarrow C)$' and '$(A \rightarrow (B \rightarrow C))$' do not have the same truth-table

A	B	C	$(A \rightarrow B) \rightarrow C$	$A \rightarrow (B \rightarrow C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **F** T F F
T	F	T	T F F **T** T	T **T** F T T
T	F	F	T F F **T** F	T **T** F T F
F	T	T	F T T **T** T	F **T** T T T
F	T	F	F T T **F** F	F **T** T F F
F	F	T	F T F **T** T	F **T** F T T
F	F	F	F T F **F** F	F **T** F T F

Also, check whether:

5. '$((A \leftrightarrow B) \leftrightarrow C)$' and '$(A \leftrightarrow (B \leftrightarrow C))$' have the same truth-table
Indeed they do:

A	B	C	$(A \leftrightarrow B) \leftrightarrow C$	$A \leftrightarrow (B \leftrightarrow C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **F** T F F
T	F	T	T F F **F** T	T **F** F F T
T	F	F	T F F **T** F	T **T** F T F
F	T	T	F F T **F** T	F **F** T T T
F	T	F	F F T **T** F	F **T** T F F
F	F	T	F T F **T** T	F **T** F F T
F	F	F	F T F **F** F	F **F** F T F

Semantic concepts 11

A. Revisit your answers to §10**A**. Determine which sentences were tautologies, which were tautological contradictions, and which were neither tautologies nor tautological contradictions.

1. $A \rightarrow A$ Tautology
2. $C \rightarrow \neg C$ Neither
3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$ Tautology
4. $(A \rightarrow B) \vee (B \rightarrow A)$ Tautology
5. $(A \wedge B) \rightarrow (B \vee A)$ Tautology
6. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ Tautology
7. $\big[(A \wedge B) \wedge \neg(A \wedge B)\big] \wedge C$ Tautological contradiction
8. $[(A \wedge B) \wedge C] \rightarrow B$ Tautology
9. $\neg\big[(C \vee A) \vee B\big]$ Neither

B. Use truth-tables to determine whether these sentences are jointly tautologically consistent, or jointly tautologically inconsistent:

1. $A \rightarrow A$, $\neg A \rightarrow \neg A$, $A \wedge A$, $A \vee A$ Jointly tautologically consistent (see line 1)

A	$A \rightarrow A$	$\neg A \rightarrow \neg A$	$A \wedge A$	$A \vee A$
T	T **T** T	F T **T** F T	T **T** T	T **T** T
F	F **T** F	T F **T** T F	F **F** F	F **F** F

2. $A \vee B$, $A \rightarrow C$, $B \rightarrow C$ Jointly tautologically consistent (see line 1)

A	B	C	$A \vee B$	$A \rightarrow C$	$B \rightarrow C$
T	T	T	T **T** T	T **T** T	T **T** T
T	T	F	T **T** T	T **F** F	T **F** F
T	F	T	T **T** F	T **T** T	F **T** T
T	F	F	T **T** F	T **F** F	F **T** F
F	T	T	F **T** T	F **T** T	T **T** T
F	T	F	F **T** T	F **T** F	T **F** F
F	F	T	F **F** F	F **T** T	F **T** T
F	F	F	F **F** F	F **T** F	F **T** F

3. $B \wedge (C \vee A)$, $A \rightarrow B$, $\neg(B \vee C)$ Jointly tautologically inconsistent

A	B	C	$B \wedge (C \vee A)$	$A \rightarrow B$	$\neg (B \vee C)$
T	T	T	T **T** T T T	T **T** T	**F** T T T
T	T	F	T **T** F T T	T **T** T	**F** T T F
T	F	T	F **F** T T T	T **F** F	**F** F T T
T	F	F	F **F** F T T	T **F** F	**T** F F F
F	T	T	T **T** T T F	F **T** T	**F** T T T
F	T	F	T **F** F F F	F **T** T	**F** T T F
F	F	T	F **F** T T F	F **T** F	**F** F T T
F	F	F	F **F** F F F	F **T** F	**T** F F F

4. $A \leftrightarrow (B \vee C)$, $C \rightarrow \neg A$, $A \rightarrow \neg B$ Jointly tautologically consistent (see line 8)

A	B	C	$A \leftrightarrow (B \vee C)$	$C \rightarrow \neg A$	$A \rightarrow \neg B$
T	T	T	T **T** T T T	T **F** F T	T **F** F T
T	T	F	T **T** T T F	F **T** F T	T **F** F T
T	F	T	T **T** F T T	T **F** F T	T **T** T F
T	F	F	T **F** F F F	F **T** F T	T **T** T F
F	T	T	F **F** T T T	T **T** T F	F **T** F T
F	T	F	F **F** T T F	F **T** T F	F **T** F T
F	F	T	F **F** F T T	T **T** T F	F **T** T F
F	F	F	F **T** F F F	F **T** T F	F **T** T F

C. Use truth-tables to determine whether each argument is tautologically valid or tautologically invalid.

1. $A \rightarrow A \therefore A$ Tautologically invalid (see line 2)

A	$A \rightarrow A$	A
T	T **T** T	T
F	F **T** F	F

2. $A \rightarrow (A \wedge \neg A) \therefore \neg A$ Tautologially valid

A	$A \rightarrow (A \wedge \neg A)$	$\neg A$
T	T **F** T F F T	**F** T
F	F **T** F F T F	**T** F

3. $A \vee (B \rightarrow A) \therefore \neg A \rightarrow \neg B$ Tautologically valid

A	B	$A \vee (B \rightarrow A)$	$\neg A \rightarrow \neg B$
T	T	T **T** T T T	F T **T** F T
T	F	T **T** F T T	F T **T** T F
F	T	F **F** T F F	T F **F** F T
F	F	F **T** F T F	T F **T** T F

4. $A \vee B, B \vee C, \neg A \therefore B \wedge C$ Tautologically invalid (see line 6)

A	B	C	$A \vee B$	$B \vee C$	$\neg A$	$B \wedge C$
T	T	T	T **T** T	T **T** T	**F** T	T **T** T
T	T	F	T **T** T	T **T** F	**F** T	T **F** F
T	F	T	T **T** F	F **T** T	**F** T	F **F** T
T	F	F	T **T** F	F **F** F	**F** T	F **F** F
F	T	T	F **T** T	T **T** T	**T** F	T **T** T
F	T	F	F **T** T	T **T** F	**T** F	T **F** F
F	F	T	F **F** F	F **T** T	**T** F	F **F** T
F	F	F	F **F** F	F **F** F	**T** F	F **F** F

5. $(B \wedge A) \rightarrow C, (C \wedge A) \rightarrow B \therefore (C \wedge B) \rightarrow A$ Tautologically invalid (see line 5)

A	B	C	$(B \wedge A) \rightarrow C$	$(C \wedge A) \rightarrow B$	$(C \wedge B) \rightarrow A$
T	T	T	T T T **T** T	T T T **T** T	T T T **T** T
T	T	F	T T T **F** F	F F T **T** T	F F T **T** T
T	F	T	F F T **T** T	T T T **F** F	T F F **T** T
T	F	F	F F T **T** F	F F T **T** F	F F F **T** T
F	T	T	T F F **T** T	T F F **T** T	T T T **F** F
F	T	F	T F F **T** F	F F F **T** T	F F T **T** F
F	F	T	F F F **T** T	T F F **T** F	T F F **T** F
F	F	F	F F F **T** F	F F F **T** F	F F F **T** F

D. Answer each of the questions below and justify your answer.

1. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are tautologically equivalent. What can you say about $\mathcal{A} \leftrightarrow \mathcal{B}$?
   $\mathcal{A}$ and $\mathcal{B}$ have the same truth-value on every line of a complete truth-table, so $\mathcal{A} \leftrightarrow \mathcal{B}$ is true on every line. It is a tautology.
2. Suppose that $(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}$ is neither a tautology nor a tautological contradiction. What can you say about whether $\mathcal{A}, \mathcal{B} \therefore \mathcal{C}$ is tautologically valid?
   Since the sentence $(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}$ is not a tautology, there is some line on which it is false. Since it is a conditional, on that line, $\mathcal{A}$ and $\mathcal{B}$ are true and $\mathcal{C}$ is false. So the argument is tautologically invalid.
3. Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are jointly tautologically inconsistent. What can you say about $(\mathcal{A} \wedge \mathcal{B} \wedge \mathcal{C})$?
   Since the sentences are jointly tautologically inconsistent, there is no valuation on which they are all true. So their conjunction is false on every valuation. It is a tautological contradiction
4. Suppose that $\mathcal{A}$ is a tautological contradiction. What can you say about whether $\mathcal{A}, \mathcal{B} \vDash \mathcal{C}$?
   Since $\mathcal{A}$ is false on every line of a complete truth-table, there is no line on which $\mathcal{A}$ and $\mathcal{B}$ are true and $\mathcal{C}$ is false. So the entailment holds.
5. Suppose that $\mathcal{C}$ is a tautology. What can you say about whether $\mathcal{A}, \mathcal{B} \vDash \mathcal{C}$?
   Since $\mathcal{C}$ is true on every line of a complete truth-table, there is no line on which $\mathcal{A}$ and $\mathcal{B}$ are true and $\mathcal{C}$ is false. So the entailment holds.
6. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are tautologically equivalent. What can you say about $(\mathcal{A} \vee \mathcal{B})$?

Not much. Since $\mathcal{A}$ and $\mathcal{B}$ are true on exactly the same lines of the truth-table, their disjunction is true on exactly the same lines. So, their disjunction is tautologically equivalent to them.

7. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are *not* tautologically equivalent. What can you say about $(\mathcal{A} \vee \mathcal{B})$?
   $\mathcal{A}$ and $\mathcal{B}$ have different truth-values on at least one line of a complete truth-table, and $(\mathcal{A} \vee \mathcal{B})$ will be true on that line. On other lines, it might be true or false. So $(\mathcal{A} \vee \mathcal{B})$ is either a tautology or it is contingent; it is *not* a tautological contradiction.

E. Consider the following principle:

- Suppose $\mathcal{A}$ and $\mathcal{B}$ are tautologically equivalent. Suppose an argument contains $\mathcal{A}$ (either as a premise, or as the conclusion). The tautological validity of the argument would be unaffected, if we replaced $\mathcal{A}$ with $\mathcal{B}$.

Is this principle correct? Explain your answer.
The principle is correct. Since $\mathcal{A}$ and $\mathcal{B}$ are tautologically equivalent, they have the same truth-table. So every valuation that makes $\mathcal{A}$ true also makes $\mathcal{B}$ true, and every valuation that makes $\mathcal{A}$ false also makes $\mathcal{B}$ false. So if no valuation makes all the premises true and the conclusion false, when $\mathcal{A}$ was among the premises or the conclusion, then no valuation makes all the premises true and the conclusion false, when we replace $\mathcal{A}$ with $\mathcal{B}$.

Truth-table shortcuts 12

A. Determine whether each sentence is a tautology, a tautological contradiction, or neither. Feel free to use shortcuts, if you would like!

1. $\neg B \wedge B$ — Tautological contradiction

B	$\neg$	B	$\wedge$	B
T	F		**F**	
F			**F**	

2. $\neg D \vee D$ — Tautology

D	$\neg$	D	$\vee$	D
T			**T**	
F	T		**T**	

3. $(A \wedge B) \vee (B \wedge A)$ — Neither

A	B	$(A$	$\wedge$	$B)$	$\vee$	$(B$	$\wedge$	$A)$
T	T		T		**T**			
T	F		F		**F**		F	
F	T		F		**F**		F	
F	F		F		**F**		F	

4. $\neg[A \rightarrow (B \rightarrow A)]$ — Tautological contradiction

A	B	$\neg[$	A	$\rightarrow$	$(B$	$\rightarrow$	$A)]$
T	T	**F**		T		T	
T	F	**F**		T		T	
F	T	**F**		T			
F	F	**F**		T			

5. $A \leftrightarrow [A \rightarrow (B \wedge \neg B)]$ — Tautological contradiction

A	B	A	$\leftrightarrow$	$[A$	$\rightarrow$	$(B$	$\wedge$	$\neg$	$B)]$
T	T		**F**		F		F	F	
T	F		**F**		F		F		
F	T		**F**		T				
F	F		**F**		T				

6. $\neg(A \wedge B) \leftrightarrow A$ — Neither

A	B	$\neg(A\wedge B)\leftrightarrow A$		
T	T	F	T	**F**
T	F	T	F	**T**
F	T	T	F	**F**
F	F	T	F	**F**

7. $A \rightarrow (B \vee C)$ Neither

A	B	C	$A\rightarrow(B\vee C)$	
T	T	T	**T**	T
T	T	F	**T**	T
T	F	T	**T**	T
T	F	F	**F**	F
F	T	T	**T**	
F	T	F	**T**	
F	F	T	**T**	
F	F	F	**T**	

8. $(A \wedge \neg A) \rightarrow (B \vee C)$ Tautology

A	B	C	$(A\wedge\neg A)\rightarrow(B\vee C)$		
T	T	T	F	F	**T**
T	T	F	F	F	**T**
T	F	T	F	F	**T**
T	F	F	F	F	**T**
F	T	T	F		**T**
F	T	F	F		**T**
F	F	T	F		**T**
F	F	F	F		**T**

9. $(B \wedge D) \leftrightarrow [A \leftrightarrow (A \vee C)]$ Neither

A	B	C	D	$(B\wedge D)\leftrightarrow[A\leftrightarrow(A\vee C)]$			
T	T	T	T	T	**T**	T	T
T	T	T	F	F	**F**	T	T
T	T	F	T	T	**T**	T	T
T	T	F	F	F	**F**	T	T
T	F	T	T	F	**F**	T	T
T	F	T	F	F	**F**	T	T
T	F	F	T	F	**F**	T	T
T	F	F	F	F	**F**	T	T
F	T	T	T	T	**F**	F	T
F	T	T	F	F	**T**	F	T
F	T	F	T	T	**T**	T	F
F	T	F	F	F	**F**	T	F
F	F	T	T	F	**T**	F	T
F	F	T	F	F	**T**	F	T
F	F	F	T	F	**F**	T	F
F	F	F	F	F	**F**	T	F

B. Determine whether these pairs of sentences are tautologically equivalent. Feel free to use shortcuts, if you would like!

1. A, $\neg A$ — Not tautologically equivalent

A	$\neg A$
T	F
F	T

2. A, $A \vee A$ — Tautologically equivalent

A	$A \vee A$
T	T
F	F

3. $A \to A$, $A \leftrightarrow A$ — Tautologically equivalent

A	$A \to A$	$A \leftrightarrow A$
T	T	T
F	T	T

4. $A \vee \neg B$, $A \to B$ — Not tautologically equivalent

A	B	$A \vee \neg B$	$A \to B$
T	T		
T	F	T	F
F	T		
F	F		

5. $A \wedge \neg A$, $\neg B \leftrightarrow B$ — Tautologically equivalent

A	B	$A \wedge \neg A$	$\neg B \leftrightarrow B$
T	T	**F** F	F **F**
T	F	**F** F	T **F**
F	T	**F**	F **F**
F	F	**F**	T **F**

6. $\neg(A \wedge B)$, $\neg A \vee \neg B$ — Tautologically equivalent

A	B	$\neg\,(A \wedge B)$	$\neg A \vee \neg B$
T	T	**F** T	F **F** F
T	F	**T** F	F **T** T
F	T	**T** F	T **T** F
F	F	**T** F	T **T** T

7. $\neg(A \to B)$, $\neg A \to \neg B$ — Not tautologically equivalent

A	B	$\neg\,(A \to B)$	$\neg A \to \neg B$
T	T	**F** T	F **T** F
T	F		
F	T		
F	F		

8. $(A \to B)$, $(\neg B \to \neg A)$ — Tautologically equivalent

A	B	$(A \to B)$	$(\neg B \to \neg A)$
T	T	T	F **T**
T	F	F	T **F** F
F	T	T	F **T**
F	F	T	T **T** T

C. Determine whether these sentences are jointly tautologically consistent, or jointly tautologically inconsistent. Feel free to use shortcuts, if you would like!

1. $A \wedge B$, $C \to \neg B$, C — Jointly tautologically inconsistent

A	B	C	$A \wedge B$	$C \to \neg B$	C
T	T	T	T	**F** F	T
T	T	F	T	**T**	F
T	F	T	F	**T** T	T
T	F	F	F	**T**	F
F	T	T	F	**F** F	T
F	T	F	F	**T**	F
F	F	T	F	**T** T	T
F	F	F	F	**T**	F

2. $A \to B$, $B \to C$, A, $\neg C$ — Jointly tautologically inconsistent

A	B	C	$A \to B$	$B \to C$	A	$\neg C$
T	T	T	T	T	T	F
T	T	F	T	F	T	T
T	F	T	F	T	T	F
T	F	F	F	T	T	T
F	T	T	T	T	F	F
F	T	F	T	F	F	T
F	F	T	T	T	F	F
F	F	F	T	T	F	T

3. $A \vee B$, $B \vee C$, $C \to \neg A$ — Jointly tautologically consistent

A	B	C	$A \vee B$	$B \vee C$	$C \to \neg A$
T	T	T			
T	T	F	T	T	T
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

D. Determine whether these arguments are tautologically valid. Feel free to use shortcuts, if you would like!

1. $A \vee \big[A \to (A \leftrightarrow A)\big] \therefore A$ Invalid

A	A	$\vee$	$[A$	$\to$	$(A \leftrightarrow A)]$	A
T						
F		**T**		T		F

2. $A \leftrightarrow \neg(B \leftrightarrow A) \therefore A$ Invalid

A	B	A	$\leftrightarrow$	$\neg$	$(B$	$\leftrightarrow$	$A)$	A
T	T							
T	F							
F	T							
F	F		**T**	F		T		F

3. $A \to B, B \therefore A$ Invalid

A	B	$A \to B$	B	A
T	T			
T	F			
F	T	T	T	F
F	F			

4. $A \vee B, B \vee C, \neg B \therefore A \wedge C$ Valid

A	B	C	$A \vee B$	$B \vee C$	$\neg B$	$A \wedge C$
T	T	T				T
T	T	F			F	F
T	F	T				T
T	F	F		F	T	F
F	T	T			F	F
F	T	F			F	F
F	F	T	F		T	F
F	F	F	F		T	F

5. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$ Valid

A	B	C	$A \leftrightarrow B$	$B \leftrightarrow C$	$A \leftrightarrow C$
T	T	T			T
T	T	F		F	F
T	F	T			T
T	F	F	F		F
F	T	T	F		F
F	T	F			T
F	F	T		F	F
F	F	F			T

Basic rules for TFL 14

A. The following two 'proofs' are *incorrect*. Explain the mistakes they make.

1	$\neg L \to (A \wedge L)$	
2	$\quad \neg L$	
3	$\quad A$	$\to$E 1, 2
4	$\quad L$	
5	$\quad \bot$	$\bot$I 4, 2
6	$\quad A$	$\bot$E 5
7	A	TND 2–3, 4–6

$\to$E on line 3 should yield '$A \wedge L$'. 'A' could then be obtained by $\wedge$E.
$\bot$I on line 5 illicitly refers to a line from a closed subproof (line 2).

1	$A \wedge (B \wedge C)$	
2	$(B \vee C) \to D$	
3	B	$\wedge$E 1
4	$B \vee C$	$\vee$I 3
5	D	$\to$E 4, 2

$\wedge$E on line 3 should yield '$B \wedge C$'. 'B' could then be obtained by $\wedge$E again.
The citation for line 5 is the wrong way round: it should be '$\to$E 2, 4'.

B. The following three proofs are missing their citations (rule and line numbers). Add them, to turn them into bona fide proofs. Additionally, write down the argument that corresponds to each proof.

1	$P \wedge S$	
2	$S \to R$	
3	P	$\wedge$E 1
4	S	$\wedge$E 1
5	R	$\to$E 2, 4
6	$R \vee E$	$\vee$I 5

Corresponding argument:
$P \wedge S, S \to R \therefore R \vee E$

1	$A \to D$	
2	$\quad A \wedge B$	
3	$\quad A$	$\wedge$E 2
4	$\quad D$	$\to$E 1, 3
5	$\quad D \vee E$	$\vee$I 4
6	$(A \wedge B) \to (D \vee E)$	$\to$I 2–5

Corresponding argument:
$A \to D \therefore (A \wedge B) \to (D \vee E)$

1	$\neg L \to (J \lor L)$	
2	$\neg L$	
3	$J \lor L$	$\to$E 1, 2
4	$\quad J$	
5	$\quad J \land J$	$\land$I 4, 4
6	$\quad J$	$\land$E 5
7	$\quad L$	
8	$\quad \bot$	$\bot$I 7, 2
9	$\quad J$	$\bot$E 8
10	J	$\lor$E 3, 4–6, 7–9

Corresponding argument: $\neg L \to (J \lor L), \neg L \therefore J$

C. Give a proof for each of the following arguments:

1. $P \therefore \neg\neg P$

1	P	
2	$\quad \neg P$	
3	$\quad \bot$	$\bot$I 1, 2
4	$\neg\neg P$	$\neg$I 2–3

2. $J \to \neg J \therefore \neg J$

1	$J \to \neg J$	
2	$\quad J$	
3	$\quad \neg J$	$\to$E 1, 2
4	$\quad \bot$	$\bot$I 2, 3
5	$\neg J$	$\neg$I 2–4

3. $Q \to (Q \land \neg Q) \therefore \neg Q$

1	$Q \to (Q \land \neg Q)$	
2	$\quad Q$	
3	$\quad Q \land \neg Q$	$\to$E 1, 2
4	$\quad \neg Q$	$\land$E 3
5	$\quad \bot$	$\bot$I 2, 4
6	$\neg Q$	$\neg$I 2–5

4. $A \to (B \to C) \therefore (A \wedge B) \to C$

1	$A \to (B \to C)$	
2	$\quad A \wedge B$	
3	$\quad A$	$\wedge$E 2
4	$\quad B \to C$	$\to$E 1, 3
5	$\quad B$	$\wedge$E 2
6	$\quad C$	$\to$E 4, 5
7	$(A \wedge B) \to C$	$\to$I 2–6

5. $K \wedge L \therefore K \leftrightarrow L$

1	$K \wedge L$	
2	$\quad K$	
3	$\quad L$	$\wedge$E 1
4	$\quad L$	
5	$\quad K$	$\wedge$E 1
6	$K \leftrightarrow L$	$\leftrightarrow$I 2–3, 4–5

6. $(C \wedge D) \vee E \therefore E \vee D$

1	$(C \wedge D) \vee E$	
2	$\quad C \wedge D$	
3	$\quad D$	$\wedge$E 2
4	$\quad E \vee D$	$\vee$I 3
5	$\quad E$	
6	$\quad E \vee D$	$\vee$I 5
7	$E \vee D$	$\vee$E 1, 2–4, 5–6

7. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$

1	$A \leftrightarrow B$	
2	$B \leftrightarrow C$	
3	$\quad A$	
4	$\quad B$	$\leftrightarrow$E 1, 3
5	$\quad C$	$\leftrightarrow$E 2, 4
6	$\quad C$	
7	$\quad B$	$\leftrightarrow$E 2, 6
8	$\quad A$	$\leftrightarrow$E 1, 7
9	$A \leftrightarrow C$	$\leftrightarrow$I 3–5, 6–8

8. $\neg F \to G, F \to H \therefore G \vee H$

1	$\neg F \to G$	
2	$F \to H$	
3	$\quad F$	
4	$\quad H$	$\to$E 2, 3
5	$\quad G \vee H$	$\vee$I 4
6	$\quad \neg F$	
7	$\quad G$	$\to$E 1, 6
8	$\quad G \vee H$	$\vee$I 7
9	$G \vee H$	TND 3–5, 6–8

9. $(Z \wedge K) \vee (K \wedge M), K \to D \therefore D$

1	$(Z \wedge K) \vee (K \wedge M)$	
2	$K \to D$	
3	$\quad Z \wedge K$	
4	$\quad K$	$\wedge$E 3
5	$\quad K \wedge M$	
6	$\quad K$	$\wedge$E 5
7	K	$\vee$E 1, 3–4, 5–6
8	D	$\to$E 2, 7

10. $P \wedge (Q \vee R), P \to \neg R \therefore Q \vee E$

1	$P \wedge (Q \vee R)$	
2	$P \to \neg R$	
3	P	$\wedge$E 1
4	$\neg R$	$\to$E 2, 3
5	$Q \vee R$	$\wedge$E 1
6	$\quad Q$	
7	$\quad Q \vee E$	$\vee$I 6
8	$\quad R$	
9	$\quad \bot$	$\bot$I 8, 4
10	$\quad Q \vee E$	$\bot$E 9
11	$Q \vee E$	$\vee$E 5, 6–7, 8–10

11. $S \leftrightarrow T \therefore S \leftrightarrow (T \vee S)$

1	$S \leftrightarrow T$	
2	$\quad S$	
3	$\quad T$	$\leftrightarrow$E 1, 2
4	$\quad T \vee S$	$\vee$I 3
5	$\quad T \vee S$	
6	$\quad\quad T$	
7	$\quad\quad S$	$\leftrightarrow$E 1, 6
8	$\quad\quad S$	
9	$\quad\quad S \wedge S$	$\wedge$I 8, 8
10	$\quad\quad S$	$\wedge$E 9
11	$\quad S$	$\vee$E 5, 6–7, 8–10
12	$S \leftrightarrow (T \vee S)$	$\leftrightarrow$I 2–4, 5–11

12. $\neg(P \rightarrow Q) \therefore \neg Q$

1	$\neg(P \rightarrow Q)$	
2	$\quad Q$	
3	$\quad\quad P$	
4	$\quad\quad Q \wedge Q$	$\wedge$I 2, 2
5	$\quad\quad Q$	$\wedge$E 4
6	$\quad P \rightarrow Q$	$\rightarrow$I 3–5
7	$\quad \bot$	$\bot$I 6, 1
8	$\neg Q$	$\neg$I 2–7

13. $\neg(P \rightarrow Q) \therefore P$

1	$\neg(P \rightarrow Q)$	
2	$\quad P$	
3	$\quad P \wedge P$	$\wedge$I 2, 2
4	$\quad P$	$\wedge$E 3
5	$\quad \neg P$	
6	$\quad\quad P$	
7	$\quad\quad \bot$	$\bot$I 6, 5
8	$\quad\quad Q$	$\bot$E 7
9	$\quad P \rightarrow Q$	$\rightarrow$I 6–8
10	$\quad \bot$	$\bot$I 9, 1
11	$\quad P$	$\bot$E 10
12	P	TND 2–4, 5–11

Additional rules for TFL 15

A. The following proofs are missing their citations (rule and line numbers). Add them wherever they are required:

1	$W \to \neg B$	
2	$A \wedge W$	
3	$B \vee (J \wedge K)$	
4	W	$\wedge$E 2
5	$\neg B$	$\to$E 1, 4
6	$J \wedge K$	DS 3, 5
7	K	$\wedge$E 6

1	$L \leftrightarrow \neg O$	
2	$L \vee \neg O$	
3	$\quad \neg L$	
4	$\quad \neg O$	DS 2, 3
5	$\quad L$	$\leftrightarrow$E 1, 4
6	$\quad \bot$	$\bot$I 5, 3
7	$\neg\neg L$	$\neg$I 3–6
8	L	DNE 7

1	$Z \to (C \wedge \neg N)$	
2	$\neg Z \to (N \wedge \neg C)$	
3	$\quad \neg(N \vee C)$	
4	$\quad \neg N \wedge \neg C$	DeM 3
5	$\quad \neg N$	$\wedge$E 4
6	$\quad \neg C$	$\wedge$E 4
7	$\quad\quad Z$	
8	$\quad\quad C \wedge \neg N$	$\to$E 1, 7
9	$\quad\quad C$	$\wedge$E 8
10	$\quad\quad \bot$	$\bot$I 9, 6
11	$\quad \neg Z$	$\neg$I 7–10
12	$\quad N \wedge \neg C$	$\to$E 2, 11
13	$\quad N$	$\wedge$E 12
14	$\quad \bot$	$\bot$I 13, 5
15	$\neg\neg(N \vee C)$	$\neg$I 3–14
16	$N \vee C$	DNE 15

B. Give a proof for each of these arguments:

1. $E \vee F$, $F \vee G$, $\neg F \therefore E \wedge G$

1	$E \vee F$	
2	$F \vee G$	
3	$\neg F$	
4	E	DS 1, 3
5	G	DS 2, 3
6	$E \wedge G$	$\wedge$I 4, 5

2. $M \vee (N \rightarrow M) \therefore \neg M \rightarrow \neg N$

1	$M \vee (N \rightarrow M)$	
2	$\quad \neg M$	
3	$\quad N \rightarrow M$	DS 1, 2
4	$\quad \neg N$	MT 3, 2
5	$\neg M \rightarrow \neg N$	$\rightarrow$I 2–4

3. $(M \vee N) \wedge (O \vee P)$, $N \rightarrow P$, $\neg P \therefore M \wedge O$

1	$(M \vee N) \wedge (O \vee P)$	
2	$N \rightarrow P$	
3	$\neg P$	
4	$\neg N$	MT 2, 3
5	$M \vee N$	$\wedge$E 1
6	M	DS 5, 4
7	$O \vee P$	$\wedge$E 1
8	O	DS 7, 3
9	$M \wedge O$	$\wedge$I 6, 8

4. $(X \wedge Y) \vee (X \wedge Z)$, $\neg(X \wedge D)$, $D \vee M \therefore M$

1	$(X \wedge Y) \vee (X \wedge Z)$	
2	$\neg(X \wedge D)$	
3	$D \vee M$	
4	$\quad X \wedge Y$	
5	$\quad X$	$\wedge$E 4
6	$\quad X \wedge Z$	
7	$\quad X$	$\wedge$E 6
8	X	$\vee$E 1, 4–5, 6–7
9	$\quad D$	
10	$\quad X \wedge D$	$\wedge$I 8, 9
11	$\quad \bot$	$\bot$I 10, 2
12	$\neg D$	$\neg$I 9–11
13	M	DS 3, 12

Derived rules 17

A. Provide proof schemes that justify the addition of the third and fourth De Morgan rules as derived rules.

Third rule:

m	$\neg\mathcal{A} \wedge \neg\mathcal{B}$	
k	$\neg\mathcal{A}$	$\wedge$E m
$k+1$	$\neg\mathcal{B}$	$\wedge$E m
$k+2$	$\quad\mathcal{A} \vee \mathcal{B}$	
$k+3$	$\quad\quad\mathcal{A}$	
$k+4$	$\quad\quad\bot$	$\bot$I $k+3$, k
$k+5$	$\quad\quad\mathcal{B}$	
$k+6$	$\quad\quad\bot$	$\bot$I $k+5$, $k+1$
$k+7$	$\quad\bot$	$\vee$E $k+2$, $k+3$–$k+4$, $k+5$–$k+6$
$k+8$	$\neg(\mathcal{A} \vee \mathcal{B})$	$\neg$I $k+2$–$k+7$

Fourth rule:

m	$\neg(\mathcal{A} \vee \mathcal{B})$	
k	$\quad\mathcal{A}$	
$k+1$	$\quad\mathcal{A} \vee \mathcal{B}$	$\vee$I k
$k+2$	$\quad\bot$	$\bot$I $k+1$, m
$k+3$	$\neg\mathcal{A}$	$\neg$I k–$k+2$
$k+4$	$\quad\mathcal{B}$	
$k+5$	$\quad\mathcal{A} \vee \mathcal{B}$	$\vee$I $k+4$
$k+6$	$\quad\bot$	$\bot$I $k+5$, m
$k+7$	$\neg\mathcal{B}$	$\neg$I $k+4$–$k+6$
$k+8$	$\neg\mathcal{A} \wedge \neg\mathcal{B}$	$\wedge$I $k+3$, $k+7$

Proof-theoretic concepts 18

A. Show that each of the following sentences is a theorem:

1. $O \to O$

1	$\quad \mid O$	
2	$\quad \mid O$	R 1
3	$O \to O$	$\to$I 1–2

2. $N \lor \neg N$

1	$\quad \mid N$	
2	$\quad \mid N \lor \neg N$	$\lor$I 1
3	$\quad \mid \neg N$	
4	$\quad \mid N \lor \neg N$	$\lor$I 3
5	$N \lor \neg N$	TND 1–2, 3–4

3. $J \leftrightarrow [J \lor (L \land \neg L)]$

1	$\quad \mid J$	
2	$\quad \mid J \lor (L \land \neg L)$	$\lor$I 1
3	$\quad \mid J \lor (L \land \neg L)$	
4	$\quad \mid \quad \mid L \land \neg L$	
5	$\quad \mid \quad \mid L$	$\land$E 4
6	$\quad \mid \quad \mid \neg L$	$\land$E 4
7	$\quad \mid \quad \mid \bot$	$\bot$I 5, 6
8	$\quad \mid \neg(L \land \neg L)$	$\neg$I 4–7
9	$\quad \mid J$	DS 3, 8
10	$J \leftrightarrow [J \lor (L \land \neg L)]$	$\leftrightarrow$I 1–2, 3–9

4. $((A \to B) \to A) \to A$

1	$\mid \mid (A \to B) \to A$	
2	$\mid \mid \mid \neg A$	
3	$\mid \mid \mid \neg(A \to B)$	MT 1, 2
4	$\mid \mid \mid \mid A$	
5	$\mid \mid \mid \mid \bot$	$\bot$I 4, 2
6	$\mid \mid \mid \mid B$	$\bot$E 5
7	$\mid \mid \mid A \to B$	$\to$I 4–6
8	$\mid \mid \mid \bot$	$\bot$I 7, 3
9	$\mid \mid \neg\neg A$	$\neg$I 2
10	$\mid \mid A$	DNE 9
11	$\mid ((A \to B) \to A) \to A$	$\to$I 1–10

B. Provide proofs to show each of the following:

1. $C \to (E \wedge G), \neg C \to G \vdash G$

1	$\mid C \to (E \wedge G)$	
2	$\mid \neg C \to G$	
3	$\mid \mid C$	
4	$\mid \mid E \wedge G$	$\to$E 1, 3
5	$\mid \mid G$	$\wedge$E 4
6	$\mid \mid \neg C$	
7	$\mid \mid G$	$\to$E 2, 6
8	$\mid G$	TND 3–5, 6–7

2. $M \wedge (\neg N \to \neg M) \vdash (N \wedge M) \vee \neg M$

1	$\mid M \wedge (\neg N \to \neg M)$	
2	$\mid M$	$\wedge$E 1
3	$\mid \neg N \to \neg M$	$\wedge$E 1
4	$\mid \mid \neg N$	
5	$\mid \mid \neg M$	$\to$E 3, 4
6	$\mid \mid \bot$	$\bot$I 2, 5
7	$\mid \neg\neg N$	$\neg$I 4–6
8	$\mid N$	DNE 7
9	$\mid N \wedge M$	$\wedge$I 8, 2
10	$\mid (N \wedge M) \vee \neg M$	$\vee$I 9

3. $(Z \wedge K) \leftrightarrow (Y \wedge M), D \wedge (D \rightarrow M) \vdash Y \rightarrow Z$

1	$(Z \wedge K) \leftrightarrow (Y \wedge M)$	
2	$D \wedge (D \rightarrow M)$	
3	D	$\wedge$E 2
4	$D \rightarrow M$	$\wedge$E 2
5	M	$\rightarrow$E 4, 3
6	$\quad Y$	
7	$\quad Y \wedge M$	$\wedge$I 6, 5
8	$\quad Z \wedge K$	$\leftrightarrow$E 1, 7
9	$\quad Z$	$\wedge$E 8
10	$Y \rightarrow Z$	$\rightarrow$I 6–9

4. $(W \vee X) \vee (Y \vee Z), X \rightarrow Y, \neg Z \vdash W \vee Y$

1	$(W \vee X) \vee (Y \vee Z)$	
2	$X \rightarrow Y$	
3	$\neg Z$	
4	$\quad W \vee X$	
5	$\quad\quad W$	
6	$\quad\quad W \vee Y$	$\vee$I 5
7	$\quad\quad X$	
8	$\quad\quad Y$	$\rightarrow$E 2, 7
9	$\quad\quad W \vee Y$	$\vee$I 8
10	$\quad W \vee Y$	$\vee$E 4, 5–6, 7–9
11	$\quad Y \vee Z$	
12	$\quad Y$	DS 11, 3
13	$\quad W \vee Y$	$\vee$I 12
14	$W \vee Y$	$\vee$E 1, 4–10, 11–13

C. Show that each of the following pairs of sentences are provably equivalent:

1. $R \leftrightarrow E$, $E \leftrightarrow R$

1	$R \leftrightarrow E$	
2	$\quad E$	
3	$\quad R$	$\leftrightarrow$E 1, 2
4	$\quad R$	
5	$\quad E$	$\leftrightarrow$E 1, 4
6	$E \leftrightarrow R$	$\leftrightarrow$I 2–3, 4–5

1	$E \leftrightarrow R$	
2	$\quad E$	
3	$\quad R$	$\leftrightarrow$E 1, 2
4	$\quad R$	
5	$\quad E$	$\leftrightarrow$E 1, 4
6	$R \leftrightarrow E$	$\leftrightarrow$I 4–5, 2–3

2. G, $\neg\neg\neg\neg G$

1	G	
2	$\mid\ \neg\neg\neg G$	
3	$\mid\ \neg G$	DNE 2
4	$\mid\ \bot$	$\bot$I 1, 3
5	$\neg\neg\neg\neg G$	$\neg$I 2–4

1	$\neg\neg\neg\neg G$	
2	$\neg\neg G$	DNE 1
3	G	DNE 2

3. $T \rightarrow S$, $\neg S \rightarrow \neg T$

1	$T \rightarrow S$	
2	$\mid\ \neg S$	
3	$\mid\ \neg T$	MT 1, 2
4	$\neg S \rightarrow \neg T$	$\rightarrow$I 2–3

1	$\neg S \rightarrow \neg T$	
2	$\mid\ T$	
3	$\mid\ \mid\ \neg S$	
4	$\mid\ \mid\ \neg T$	$\rightarrow$E 1, 3
5	$\mid\ \mid\ \bot$	$\bot$I 2, 4
6	$\mid\ \neg\neg S$	$\neg$I 3–5
7	$\mid\ S$	DNE 6
8	$T \rightarrow S$	$\rightarrow$I 2–7

4. $U \rightarrow I$, $\neg(U \wedge \neg I)$

1	$U \rightarrow I$	
2	$\mid\ U \wedge \neg I$	
3	$\mid\ U$	$\wedge$E 2
4	$\mid\ \neg I$	$\wedge$E 2
5	$\mid\ I$	$\rightarrow$E 1, 3
6	$\mid\ \bot$	$\bot$I 5, 4
7	$\neg(U \wedge \neg I)$	$\neg$I 2–6

1	$\neg(U \wedge \neg I)$	
2	$\mid\ U$	
3	$\mid\ \mid\ \neg I$	
4	$\mid\ \mid\ U \wedge \neg I$	$\wedge$I 2, 3
5	$\mid\ \mid\ \bot$	$\bot$I 4, 1
6	$\mid\ \neg\neg I$	$\neg$I 3–5
7	$\mid\ I$	DNE 6
8	$U \rightarrow I$	$\rightarrow$I 2–7

5. $\neg(C \to D), C \wedge \neg D$

1	$\mid C \wedge \neg D$	
2	$\mid C$	$\wedge$E 1
3	$\mid \neg D$	$\wedge$E 1
4	$\mid \mid C \to D$	
5	$\mid \mid D$	$\to$E 4, 2
6	$\mid \mid \bot$	$\bot$I 5, 3
7	$\mid \neg(C \to D)$	$\neg$I 4–6

1	$\mid \neg(C \to D)$	
2	$\mid \mid D$	
3	$\mid \mid \mid C$	
4	$\mid \mid \mid D$	R 2
5	$\mid \mid C \to D$	$\to$I 3–4
6	$\mid \mid \bot$	$\bot$I 5, 1
7	$\mid \neg D$	$\neg$I 2–6
8	$\mid \mid \neg C$	
9	$\mid \mid \mid C$	
10	$\mid \mid \mid \bot$	$\bot$I 9, 8
11	$\mid \mid \mid D$	$\bot$E 10
12	$\mid \mid C \to D$	$\to$I 9–11
13	$\mid \mid \bot$	$\bot$I 12, 1
14	$\mid \neg\neg C$	$\neg$I 8–13
15	$\mid C$	DNE 14
16	$\mid C \wedge \neg D$	$\wedge$I 15, 7

6. $\neg G \leftrightarrow H$, $\neg(G \leftrightarrow H)$

1	$\mid \neg G \leftrightarrow H$	
2	$\mid \mid G \leftrightarrow H$	
3	$\mid \mid \mid G$	
4	$\mid \mid \mid H$	$\leftrightarrow$E 2, 3
5	$\mid \mid \mid \neg G$	$\leftrightarrow$E 1, 4
6	$\mid \mid \mid \bot$	$\bot$I 3, 5
7	$\mid \mid \mid \neg G$	
8	$\mid \mid \mid H$	$\leftrightarrow$E 1, 7
9	$\mid \mid \mid G$	$\leftrightarrow$E 2, 8
10	$\mid \mid \mid \bot$	$\bot$I 9, 7
11	$\mid \mid \bot$	TND 3–6, 7–10
12	$\mid \neg(G \leftrightarrow H)$	$\neg$I 2–11

1	$\neg(G \leftrightarrow H)$	
2	$\quad \neg G$	
3	$\quad\quad \neg H$	
4	$\quad\quad\quad G$	
5	$\quad\quad\quad \bot$	$\bot$I 4, 2
6	$\quad\quad\quad H$	$\bot$E 5
7	$\quad\quad\quad H$	
8	$\quad\quad\quad \bot$	$\bot$I 7, 3
9	$\quad\quad\quad G$	$\bot$E 8
10	$\quad\quad G \leftrightarrow H$	$\leftrightarrow$I 4–6, 7–9
11	$\quad\quad \bot$	$\bot$I 10, 1
12	$\quad \neg\neg H$	$\neg$I 3–11
13	$\quad H$	DNE 12
14	$\quad H$	
15	$\quad\quad G$	
16	$\quad\quad\quad G$	
17	$\quad\quad\quad H$	R 14
18	$\quad\quad\quad H$	
19	$\quad\quad\quad G$	R 15
20	$\quad\quad G \leftrightarrow H$	$\leftrightarrow$I 16–17, 18–19
21	$\quad\quad \bot$	$\bot$I 20, 1
22	$\quad \neg G$	$\neg$I 15–21
23	$\neg G \leftrightarrow H$	$\leftrightarrow$I 2–13, 14–22

D. If you know that $\mathcal{A} \vdash \mathcal{B}$, what can you say about $(\mathcal{A} \wedge \mathcal{C}) \vdash \mathcal{B}$? What about $(\mathcal{A} \vee \mathcal{C}) \vdash \mathcal{B}$? Explain your answers.

If $\mathcal{A} \vdash \mathcal{B}$, then $(\mathcal{A} \wedge \mathcal{C}) \vdash \mathcal{B}$. After all, if $\mathcal{A} \vdash \mathcal{B}$, then there is some proof with assumption $\mathcal{A}$ that ends with $\mathcal{B}$, and no undischarged assumptions other than $\mathcal{A}$. Now, if we start a proof with assumption $(\mathcal{A} \wedge \mathcal{C})$, we can obtain $\mathcal{A}$ by $\wedge$E. We can now copy and paste the original proof of $\mathcal{B}$ from $\mathcal{A}$, adding 1 to every line number and line number citation. The result will be a proof of $\mathcal{B}$ from assumption $\mathcal{A}$.

However, we cannot prove much from $(\mathcal{A} \vee \mathcal{C})$. After all, it might be impossible to prove $\mathcal{B}$ from $\mathcal{C}$.

E. According to a result known as the DEDUCTION THEOREM, $\mathcal{A} \vdash \mathcal{C}$ iff $\vdash \mathcal{A} \rightarrow \mathcal{C}$. Give a demonstration of this result.

If $\mathcal{A} \vdash \mathcal{C}$, then there is a proof which has this shape:

1	$\mathcal{A}$	
	$\ldots$	
	$\ldots$	
	$\ldots$	
n	$\mathcal{C}$	

We can turn this into a proof that $\vdash \mathcal{A} \to \mathcal{C}$ just by using $\to$I to discharge the initial assumption:

1	$\quad \mathcal{A}$	
	$\quad \ldots$	
	$\quad \ldots$	
	$\quad \ldots$	
n	$\quad \mathcal{C}$	
$n+1$	$\mathcal{A} \to \mathcal{C}$	$\to$I 1–n

So if $\mathcal{A} \vdash \mathcal{C}$, then $\vdash \mathcal{A} \to \mathcal{C}$. And the converse is clearly true too. Suppose we started with the proof that $\vdash \mathcal{A} \to \mathcal{C}$. We could cut the subproof out, and take it as a free-standing proof that $\mathcal{A} \vdash \mathcal{C}$. So $\mathcal{A} \vdash \mathcal{C}$ iff $\vdash \mathcal{A} \to \mathcal{C}$.

Sentences with one quantifier 20

A. Here are the syllogistic figures identified by Aristotle and his successors, along with their medieval names:

- **Barbara.** All G are F. All H are G. So: All H are F
  $\forall x(Gx \to Fx), \forall x(Hx \to Gx) \therefore \forall x(Hx \to Fx)$
- **Celarent.** No G are F. All H are G. So: No H are F
  $\forall x(Gx \to \neg Fx), \forall x(Hx \to Gx) \therefore \forall x(Hx \to \neg Fx)$
- **Ferio.** No G are F. Some H is G. So: Some H is not F
  $\forall x(Gx \to \neg Fx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Darii.** All G are H. Some H is G. So: Some H is F.
  $\forall x(Gx \to Fx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge Fx)$
- **Camestres.** All F are G. No H are G. So: No H are F.
  $\forall x(Fx \to Gx), \forall x(Hx \to \neg Gx) \therefore \forall x(Hx \to \neg Fx)$
- **Cesare.** No F are G. All H are G. So: No H are F.
  $\forall x(Fx \to \neg Gx), \forall x(Hx \to Gx) \therefore \forall x(Hx \to \neg Fx)$
- **Baroko.** All F are G. Some H is not G. So: Some H is not F.
  $\forall x(Fx \to Gx), \exists x(Hx \wedge \neg Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Festino.** No F are G. Some H are G. So: Some H is not F.
  $\forall x(Fx \to \neg Gx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Datisi.** All G are F. Some G is H. So: Some H is F.
  $\forall x(Gx \to Fx), \exists x(Gx \wedge Hx) \therefore \exists x(Hx \wedge Fx)$
- **Disamis.** Some G is F. All G are H. So: Some H is F.
  $\exists x(Gx \wedge Fx), \forall x(Gx \to Hx) \therefore \exists x(Hx \wedge Fx)$
- **Ferison.** No G are F. Some G is H. So: Some H is not F.
  $\forall x(Gx \to \neg Fx), \exists x(Gx \wedge Hx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Bokardo.** Some G is not F. All G are H. So: Some H is not F.
  $\exists x(Gx \wedge \neg Fx), \forall x(Gx \to Hx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Camenes.** All F are G. No G are H So: No H is F.
  $\forall x(Fx \to Gx), \forall x(Gx \to \neg Hx) \therefore \forall x(Hx \to \neg Fx)$
- **Dimaris.** Some F is G. All G are H. So: Some H is F.
  $\exists x(Fx \wedge Gx), \forall x(Gx \to Hx) \therefore \exists x(Hx \wedge Fx)$
- **Fresison.** No F are G. Some G is H. So: Some H is not F.
  $\forall x(Fx \to \neg Gx), \exists x(Gx \wedge Hx) \therefore \exists(Hx \wedge \neg Fx)$

Symbolise each argument in FOL.

B. Using the following symbolisation key:

domain: people
    K: _____ knows the combination to the safe

S: _____ is a spy
V: _____ is a vegetarian
h: Hofthor
i: Ingmar

symbolise the following sentences in FOL:

1. Neither Hofthor nor Ingmar is a vegetarian.
   $\neg Vh \wedge \neg Vi$
2. No spy knows the combination to the safe.
   $\forall x(Sx \rightarrow \neg Kx)$
3. No one knows the combination to the safe unless Ingmar does.
   $\forall x \neg Kx \vee Ki$
4. Hofthor is a spy, but no vegetarian is a spy.
   $Sh \wedge \forall x(Vx \rightarrow \neg Sx)$

C. Using this symbolisation key:

domain: all animals
A: _____ is an alligator
M: _____ is a monkey
R: _____ is a reptile
Z: _____ lives at the zoo
a: Amos
b: Bouncer
c: Cleo

symbolise each of the following sentences in FOL:

1. Amos, Bouncer, and Cleo all live at the zoo.
   $Za \wedge Zb \wedge Zc$
2. Bouncer is a reptile, but not an alligator.
   $Rb \wedge \neg Ab$
3. Some reptile lives at the zoo.
   $\exists x(Rx \wedge Zx)$
4. Every alligator is a reptile.
   $\forall x(Ax \rightarrow Rx)$
5. Any animal that lives at the zoo is either a monkey or an alligator.
   $\forall x(Zx \rightarrow (Mx \vee Ax))$
6. There are reptiles which are not alligators.
   $\exists x(Rx \wedge \neg Ax)$
7. If any animal is an reptile, then Amos is.
   $\exists x Rx \rightarrow Ra$
8. If any animal is an alligator, then it is a reptile.
   $\forall x(Ax \rightarrow Rx)$

D. For each argument, write a symbolisation key and symbolise the argument in FOL.

1. Willard is a logician. All logicians wear funny hats. So Willard wears a funny hat

domain: people
L: _____ is a logician
H: _____ wears a funny hat
i: Willard

$Li, \forall x(Lx \rightarrow Hx) \therefore Hi$

2. Nothing on my desk escapes my attention. There is a computer on my desk. As such, there is a computer that does not escape my attention.

   domain: physical things
   D: _____ is on my desk
   E: _____ escapes my attention
   C: _____ is a computer

   $\forall x(Dx \rightarrow \neg Ex), \exists x(Dx \wedge Cx) \therefore \exists x(Cx \wedge \neg Ex)$

3. All my dreams are black and white. Old TV shows are in black and white. Therefore, some of my dreams are old TV shows.

   domain: episodes (psychological and televised)
   D: _____ is one of my dreams
   B: _____ is in black and white
   O: _____ is an old TV show

   $\forall x(Dx \rightarrow Bx), \forall x(Ox \rightarrow Bx) \therefore \exists x(Dx \wedge Ox)$.
   Comment: generic statements are tricky to deal with. Does the second sentence mean that *all* old TV shows are in black and white; or that most of them are; or that most of the things which are in black and white are old TV shows? I have gone with the former, but it is not clear that FOL deals with these well.

4. Neither Holmes nor Watson has been to Australia. A person could see a kangaroo only if they had been to Australia or to a zoo. Although Watson has not seen a kangaroo, Holmes has. Therefore, Holmes has been to a zoo.

   domain: people
   A: _____ has been to Australia
   K: _____ has seen a kangaroo
   Z: _____ has been to a zoo
   h: Holmes
   a: Watson

   $\neg Ah \wedge \neg Aa, \forall x(Kx \rightarrow (Ax \vee Zx)), \neg Ka \wedge Kh \therefore Zh$

5. No one expects the Spanish Inquisition. No one knows the troubles I've seen. Therefore, anyone who expects the Spanish Inquisition knows the troubles I've seen.

   domain: people
   S: _____ expects the Spanish Inquisition
   T: _____ knows the troubles I've seen

   $\forall x \neg Sx, \forall x \neg Tx \therefore \forall x(Sx \rightarrow Tx)$

6. All babies are illogical. Nobody who is illogical can manage a crocodile. Berthold is a baby. Therefore, Berthold is unable to manage a crocodile.

   domain: people
   B: _____ is a baby

I: _____ is illogical
C: _____ can manage a crocodile
b: Berthold

$$\forall x(Bx \to Ix), \forall x(Ix \to \neg Cx), Bb \therefore \neg Cb$$

Multiple generality 21

A. Using this symbolisation key:

domain: all animals
A: _____$_1$ is an alligator
M: _____$_1$ is a monkey
R: _____$_1$ is a reptile
Z: _____$_1$ lives at the zoo
L: _____$_1$ loves _____$_2$
a: Amos
b: Bouncer
c: Cleo

symbolise each of the following sentences in FOL:

1. If Cleo loves Bouncer, then Bouncer is a monkey.
   $Lcb \to Mb$
2. If both Bouncer and Cleo are alligators, then Amos loves them both.
   $(Ab \land Ac) \to (Lab \land Lac)$
3. Cleo loves a reptile.
   $\exists x(Rx \land Lcx)$
   Comment: this English expression is ambiguous; in some contexts, it can be read as a generic, along the lines of 'Cleo loves reptiles'. (Compare 'I do love a good pint'.)
4. Bouncer loves all the monkeys that live at the zoo.
   $\forall x((Mx \land Zx) \to Lbx)$
5. All the monkeys that Amos loves love him back.
   $\forall x((Mx \land Lax) \to Lxa)$
6. Every monkey that Cleo loves is also loved by Amos.
   $\forall x((Mx \land Lcx) \to Lax)$
7. There is a monkey that loves Bouncer, but sadly Bouncer does not reciprocate this love.
   $\exists x(Mx \land Lxb \land \neg Lbx)$

B. Using the following symbolisation key:

domain: all animals
C: _____$_1$ likes cartoons
D: _____$_1$ is a dog
L: _____$_1$ is larger than _____$_2$
b: Bertie

e: Emerson
f: Fergis

symbolise the following sentences in FOL:

1. Bertie is a dog who likes cartoons.
   $Db \land Cb$
2. Bertie, Emerson, and Fergis are all dogs.
   $Db \land De \land Df$
3. Emerson is larger than Bertie, and Fergis is larger than Emerson.
   $Leb \land Lfe$
4. All dogs like cartoons.
   $\forall x(Dx \to Cx)$
5. Only dogs like cartoons.
   $\forall x(Cx \to Dx)$
   Comment: the FOL sentence just written does not require that anyone likes cartoons. The English sentence might suggest that at least some dogs *do* like cartoons?
6. There is a dog that is larger than Emerson.
   $\exists x(Dx \land Lxe)$
7. If there is a dog larger than Fergis, then there is a dog larger than Emerson.
   $\exists x(Dx \land Lxf) \to \exists x(Dx \land Lxe)$
8. No animal that likes cartoons is larger than Emerson.
   $\forall x(Cx \to \neg Lxe)$
9. No dog is larger than Fergis.
   $\forall x(Dx \to \neg Lxf)$
10. Any animal that dislikes cartoons is larger than Bertie.
    $\forall x(\neg Cx \to Lxb)$
    Comment: this is very poor, though! For 'dislikes' does not mean the same as 'does not like'.
11. There is an animal that is between Bertie and Emerson in size.
    $\exists x((Lbx \land Lxe) \lor (Lex \land Lxb))$
12. There is no dog that is between Bertie and Emerson in size.
    $\forall x\big(Dx \to \neg\big[(Lbx \land Lxe) \lor (Lex \land Lxb)\big]\big)$
13. No dog is larger than itself.
    $\forall x(Dx \to \neg Lxx)$
14. Every dog is larger than some dog.
    $\forall x(Dx \to \exists y(Dy \land Lxy))$
    Comment: the English sentence is potentially ambiguous here. I have resolved the ambiguity by assuming it should be paraphrased by 'for every dog, there is a dog smaller than it'.
15. There is an animal that is smaller than every dog.
    $\exists x \forall y(Dy \to Lyx)$
16. If there is an animal that is larger than any dog, then that animal does not like cartoons.
    $\forall x(\forall y(Dy \to Lxy) \to \neg Cx)$
    Comment: I have assumed that 'larger than any dog' here means 'larger than every dog'.

C. Using the following symbolisation key:

domain: people
D: $\_\_\_\_\__1$ dances ballet
F: $\_\_\_\_\__1$ is female
M: $\_\_\_\_\__1$ is male
C: $\_\_\_\_\__1$ is a child of $\_\_\_\_\__2$
S: $\_\_\_\_\__1$ is a sibling of $\_\_\_\_\__2$
a: Abebi
n: Naija
o: Orodena

symbolise the following arguments in FOL:

1. All of Orodena's children are ballet dancers.
   $\forall x(Cxo \to Dx)$
2. Abebi is Orodena's daughter.
   $Cao \wedge Fa$
3. Orodena has a daughter.
   $\exists x(Cxo \wedge Fx)$
4. Abebi is an only child.
   $\neg\exists x Sxa$
5. All of Orodena's sons dance ballet.
   $\forall x\big[(Cxo \wedge Mx) \to Dx\big]$
6. Orodena has no sons.
   $\neg\exists x(Cxo \wedge Mx)$
7. Abebi is Naija's niece.
   $\exists x(Sxn \wedge Cax \wedge Fa)$
8. Orodena is Naija's brother.
   $Son \wedge Mo$
9. Orodena's brothers have no children.
   $\forall x\big[(Sox \wedge Mx) \to \neg\exists y Cyx\big]$
10. Abebi is an aunt.
    $Fa \wedge \exists x(Sxa \wedge \exists y Cyx)$
11. Everyone who dances ballet has a brother who also dances ballet.
    $\forall x\big[Dx \to \exists y(My \wedge Syx \wedge Dy)\big]$
12. Every female who dances ballet is the child of someone who dances ballet.
    $\forall x\big[(Fx \wedge Dx) \to \exists y(Cxy \wedge Dy)\big]$

Identity 22

A. Explain why:

- '$\exists x \forall y(Ay \leftrightarrow x = y)$' is a good symbolisation of 'there is exactly one apple'.
  We might naturally read this in English thus:
  - There is something, x, such that, if you choose any object at all, if you chose an apple then you chose x itself, and if you chose x itself then you chose an apple.

  The x in question must therefore be the one and only thing which is an apple.
- '$\exists x \exists y\big[\neg x = y \wedge \forall z\big(Az \leftrightarrow (x = z \vee y = z)\big)\big]$' is a good symbolisation of 'there are exactly two apples'.
  Similarly to the above, we might naturally read this in English thus:
  - There are two distinct things, x and y, such that if you choose any object at all, if you chose an apple then you either chose x or y, and if you chose either x or y then you chose an apple.

  The x and y in question must therefore be the only things which are apples, and since they are distinct, there are two of them.

Definite descriptions 23

A. Using the following symbolisation key:

domain: people
K: _____$_1$ knows the combination to the safe
S: _____$_1$ is a spy
V: _____$_1$ is a vegetarian
T: _____$_1$ trusts _____$_2$
h: Hofthor
i: Ingmar

symbolise the following sentences in FOL:

1. Hofthor trusts a vegetarian.
   $\exists x(Vx \wedge Thx)$
2. Everyone who trusts Ingmar trusts a vegetarian.
   $\forall x\big[Txi \rightarrow \exists y(Txy \wedge Vy)\big]$
3. Everyone who trusts Ingmar trusts someone who trusts a vegetarian.
   $\forall x\big[Txi \rightarrow \exists y\big(Txy \wedge \exists z(Tyz \wedge Vz)\big)\big]$
4. Only Ingmar knows the combination to the safe.
   $\forall x(Kx \rightarrow x = i)$
   Comment: does the English claim entail that Ingmar *does* know the combination to the safe? If so, then we should formalise this with a '$\leftrightarrow$'.
5. Ingmar trusts Hofthor, but no one else.
   $\forall x(Tix \leftrightarrow x = h)$
6. The person who knows the combination to the safe is a vegetarian.
   $\exists x\big[Kx \wedge \forall y(Ky \rightarrow x = y) \wedge Vx\big]$
7. The person who knows the combination to the safe is not a spy.
   $\exists x\big[Kx \wedge \forall y(Ky \rightarrow x = y) \wedge \neg Sx\big]$
   Comment: the scope of negation is potentially ambiguous here; I have read it as *inner* negation.

B. Using the following symbolisation key:

domain: cards in a standard deck
B: _____$_1$ is black
C: _____$_1$ is a club
D: _____$_1$ is a deuce
J: _____$_1$ is a jack
M: _____$_1$ is a man with an axe
O: _____$_1$ is one-eyed
W: _____$_1$ is wild

symbolise each sentence in FOL:

1. All clubs are black cards.
   $\forall x(Cx \rightarrow Bx)$
2. There are no wild cards.
   $\neg\exists x Wx$
3. There are at least two clubs.
   $\exists x\exists y(\neg x = y \wedge Cx \wedge Cy)$
4. There is more than one one-eyed jack.
   $\exists x\exists y(\neg x = y \wedge Jx \wedge Ox \wedge Jy \wedge Oy)$
5. There are at most two one-eyed jacks.
   $\forall x\forall y\forall z\big[(Jx \wedge Ox \wedge Jy \wedge Oy \wedge Jz \wedge Oz) \rightarrow (x = y \vee x = z \vee y = z)\big]$
6. There are two black jacks.
   $\exists x\exists y(\neg x = y \wedge Bx \wedge Jx \wedge By \wedge Jy)$
   Comment: I am reading this as 'there are *at least* two...'. If the suggestion was that there are *exactly* two, then a different FOL sentence would be required, namely:
   $\exists x\exists y\big(\neg x = y \wedge Bx \wedge Jx \wedge By \wedge Jy \wedge \forall z[(Bz \wedge Jz) \rightarrow (x = z \vee y = z)]\big)$
7. There are four deuces.
   $\exists w\exists x\exists y\exists z(\neg w = x \wedge \neg w = y \wedge \neg w = z \wedge \neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Dw \wedge Dx \wedge Dy \wedge Dz)$
   Comment: I am reading this as 'there are *at least* four...'. If the suggestion is that there are *exactly* four, then we should offer instead:
   $\exists w\exists x\exists y\exists z\big(\neg w = x \wedge \neg w = y \wedge \neg w = z \wedge \neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Dw \wedge Dx \wedge Dy \wedge Dz \wedge \forall v[Dv \rightarrow (v = w \vee v = x \vee v = y \vee v = z)]\big)$
8. The deuce of clubs is a black card.
   $\exists x\big[Dx \wedge Cx \wedge \forall y\big((Dy \wedge Cy) \rightarrow x = y\big) \wedge Bx\big]$
9. One-eyed jacks and the man with the axe are wild.
   $\forall x\big[(Jx \wedge Ox) \rightarrow Wx\big] \wedge \exists x\big[Mx \wedge \forall y(My \rightarrow x = y) \wedge Wx\big]$
10. If the deuce of clubs is wild, then there is exactly one wild card.
    $\exists x\big(Dx \wedge Cx \wedge \forall y\big[(Dy \wedge Cy) \rightarrow x = y\big] \wedge Wx\big) \rightarrow \exists x\big(Wx \wedge \forall y(Wy \rightarrow x = y)\big)$
    Comment: if there is not exactly one deuce of clubs, then the above sentence is true. Maybe that's the wrong verdict. Perhaps the sentence should definitely be taken to imply that there is one and only one deuce of clubs, and then express a conditional about wildness. If so, then we might symbolise it thus:
    $\exists x\big(Dx \wedge Cx \wedge \forall y\big[(Dy \wedge Cy) \rightarrow x = y\big] \wedge \big[Wx \rightarrow \forall y(Wy \rightarrow x = y)\big]\big)$
11. The man with the axe is not a jack.
    $\exists x\big[Mx \wedge \forall y(My \rightarrow x = y) \wedge \neg Jx\big]$
12. The deuce of clubs is not the man with the axe.
    $\exists x\exists y\big(Dx \wedge Cx \wedge \forall z[(Dz \wedge Cz) \rightarrow x = z] \wedge My \wedge \forall z(Mz \rightarrow y = z) \wedge \neg x = y\big)$

C. Using the following symbolisation key:

domain: animals in the world
- B: $\_\_\_\__1$ is in Farmer Brown's field
- H: $\_\_\_\__1$ is a horse
- C: $\_\_\_\__1$ is a cow
- F: $\_\_\_\__1$ is faster than $\_\_\_\__2$

r: Redrum

symbolise the following sentences in FOL:

1. There are at least three horses in the world.
   $\exists x \exists y \exists z(\neg z = x \wedge \neg z = y \wedge \neg y = x \wedge Hx \wedge Hy \wedge Hz)$
2. There are at least three animals in the world.
   $\exists x \exists y \exists z(\neg z = x \wedge \neg z = y \wedge \neg y = x)$
3. There is more than one horse in Farmer Brown's field.
   $\exists x \exists y(\neg x = y \wedge Hx \wedge Hy \wedge Bx \wedge By)$
4. Every horse is faster than every cow.
   $\forall x \forall y\big[(Hx \wedge Cy) \rightarrow Fxy\big]$
5. Redrum is faster than every cow in Farmer Brown's field.
   $\forall x\big[(Hx \wedge Bx) \rightarrow Frx\big]$
6. There is a cow in Farmer Brown's field that is faster than a horse in Farmer Brown's field.
   $\exists x \exists y(Cx \wedge Bx \wedge Hy \wedge By \wedge Fxy)$
7. Redrum is faster than every other horse.
   $\forall x\big[(Hx \wedge \neg x = r) \rightarrow Frx\big]$
8. The fastest horse is in Farmer Brown's field.
   $\exists x\big[Hx \wedge \forall y\big((Hy \wedge \neg x = y) \rightarrow Fxy\big) \wedge Bx\big]$
9. The fastest horse in Farmer Brown's field is faster than Redrum.
   $\exists x\big[Hx \wedge Bx \wedge \forall y\big((Hy \wedge By \wedge \neg x = y) \rightarrow Fxy\big) \wedge Fxr\big]$
10. The fastest horse in Farmer Brown's field is faster than the fastest cow in the world.
    $\exists x \exists y\big[Hx \wedge Bx \wedge \forall z\big((Hz \wedge Bz \wedge \neg x = z) \rightarrow Fxz\big) \wedge Cy \wedge \forall z\big((Cz \wedge \neg y = z) \rightarrow Fyz\big) \wedge Fxy\big]$

D. In this section, we symbolised 'Jonny is the keyboardist' by '$\exists x(Kx \wedge \forall y(Ky \rightarrow x = y) \wedge x = j)$'. Two equally good symbolisations would be:

- $Kj \wedge \forall y(Ky \rightarrow y = j)$
  This sentence requires that Jonny is a keyboardist, and that Jonny alone is a keyboardist. Otherwise put, there is one and only one keyboardist, namely Jonny. Otherwise otherwise put: Jonny is the keyboardist.
- $\forall y(Ky \leftrightarrow y = j)$
  This sentence can be understood thus: Take anything you like; now, if you chose a keyboardist, then you chose Jonny; and if you chose Jonny, then you chose a keyboardist. So there is one and only one keyboardist, namely Jonny, as required.

Explain why these would be equally good symbolisations.

Sentences of FOL 24

A. Identify which variables are bound and which are free. I shall underline the bound variables, and put free variables in blue.

1. $\exists x L\underline{x}y \wedge \forall y L\underline{y}x$
2. $\forall x A\underline{x} \wedge Bx$
3. $\forall x(A\underline{x} \wedge B\underline{x}) \wedge \forall y(Cx \wedge D\underline{y})$
4. $\forall x \exists y[R\underline{xy} \rightarrow (Jz \wedge K\underline{x})] \vee Ryx$
5. $\forall x_1(Mx_2 \leftrightarrow Lx_2\underline{x_1}) \wedge \exists x_2 Lx_3\underline{x_2}$

Truth in FOL 26

A. Consider the following interpretation:

domain: Corwin, Benedict
A: Corwin, Benedict
B: Benedict
N^1:
c: Corwin

Determine whether each of the following sentences is true or false in that interpretation:

1. Bc False
2. $Ac \leftrightarrow \neg Nc$ True
3. $Nc \rightarrow (Ac \vee Bc)$ True
4. $\forall x Ax$ True
5. $\forall x \neg Bx$ False
6. $\exists x(Ax \wedge Bx)$ True
7. $\exists x(Ax \rightarrow Nx)$ False
8. $\forall x(Nx \vee \neg Nx)$ True
9. $\exists x Bx \rightarrow \forall x Ax$ True

B. Consider the following interpretation:

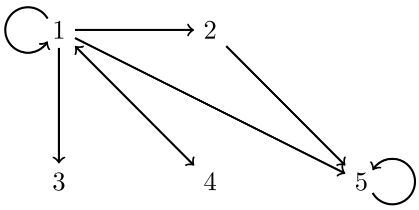
domain: Luda, Capriana, Edgar
G: Luda, Capriana, Edgar
H: Capriana
M: Luda, Edgar
c: Capriana
e: Edgar

Determine whether each of the following sentences is true or false in that interpretation:

1. Hc True
2. He False
3. $Mc \vee Me$ True
4. $Gc \vee \neg Gc$ True
5. $Mc \rightarrow Gc$ True
6. $\exists x Hx$ True
7. $\forall x Hx$ False
8. $\exists x \neg Mx$ True

9. $\exists x(Hx \wedge Gx)$ True
10. $\exists x(Mx \wedge Gx)$ True
11. $\forall x(Hx \vee Mx)$ True
12. $\exists x Hx \wedge \exists x Mx$ True
13. $\forall x(Hx \leftrightarrow \neg Mx)$ True
14. $\exists x Gx \wedge \exists x \neg Gx$ False
15. $\forall x \exists y(Gx \wedge Hy)$ True

C. Following the diagram conventions introduced at the end of §23, consider the following interpretation:



Determine whether each of the following sentences is true or false in that interpretation:

1. $\exists x Rxx$ True
2. $\forall x Rxx$ False
3. $\exists x \forall y Rxy$ True
4. $\exists x \forall y Ryx$ False
5. $\forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$ False
6. $\forall x \forall y \forall z((Rxy \wedge Rxz) \rightarrow Ryz)$ False
7. $\exists x \forall y \neg Rxy$ True
8. $\forall x(\exists y Rxy \rightarrow \exists y Ryx)$ True
9. $\exists x \exists y(\neg x = y \wedge Rxy \wedge Ryx)$ True
10. $\exists x \forall y(Rxy \leftrightarrow x = y)$ True
11. $\exists x \forall y(Ryx \leftrightarrow x = y)$ False
12. $\exists x \exists y(\neg x = y \wedge Rxy \wedge \forall z(Rzx \leftrightarrow y = z))$ True

Using interpretations 28

There are lots of right answers to the questions in this section. All of the interpretations that I will present will be highly artificial. In each case, I will use a small collection of numbers as my domain, and I will directly stipulate the extensions of the predicates. But to repeat, these are not the only right answers!

A. Show that each of the following is neither a logical truth nor a contradiction: This requires presenting two interpretations for each sentence: one which makes the sentence true, and one which makes it false.

1. $Da \wedge Db$

   An interpretation which makes sentence 1 true:

   domain: 0
   D: 0
   a: 0
   b: 0

   An interpretation which makes sentence 1 false:

   domain: 0, 1
   D: 0
   a: 0
   b: 1

2. $\exists x Txh$

   An interpretation which makes sentence 2 true:

   domain: 0
   T: <0,0>
   h: 0

An interpretation which makes sentence 2 false:

domain: 0
T:
h: 0

3. $Pm \wedge \neg\forall x Px$

An interpretation which makes sentence 3 true:

domain: 0, 1
P: 0
m: 0

An interpretation which makes sentence 3 false:

domain: 0, 1
P: 0
m: 1

4. $\forall z Jz \leftrightarrow \exists y Jy$

An interpretation which makes sentence 4 true:

domain: 0
J: 0

An interpretation which makes sentence 4 false:

domain: 0, 1
J: 0

5. $\forall x(Wxmn \vee \exists y Lxy)$

An interpretation which makes sentence 5 true:

domain: 0
W: $<0,0,0>$
L:
m: 0
n: 0

An interpretation which makes sentence 5 false:

domain: 0
W:
L:
m: 0
n: 0

6. $\exists x(Gx \rightarrow \forall y My)$

An interpretation which makes sentence 6 true:

domain: 0
G: 0
M: 0

An interpretation which makes sentence 6 false:

domain: 0
G: 0
M:

7. $\exists x(x = h \wedge x = i)$

An interpretation which makes sentence 7 true:

domain: 0
h: 0
i: 0

An interpretation which makes sentence 7 false:

domain: 0,1
h: 0
i: 1

B. Show that the following pairs of sentences are not equivalent in FOL.
To show that a pair of sentence are not equivalent in FOL, we just need to present an interpretation which makes one of the sentences true, and the other false.

1. Ja, Ka

   domain: 0
   J: 0
   K:
   a: 0

   This interpretation makes 'Ja' true and 'Ka' false.
2. $\exists xJx$, Jm

   domain: 0, 1
   J: 0
   m: 1

   This interpretation makes '$\exists xJx$' true and 'Jm' false.
3. $\forall xRxx$, $\exists xRxx$

   domain: 0, 1
   R: $<0,0>$

   This interpretation makes '$\exists xRxx$' true and '$\forall xRxx$' false.
4. $\exists xPx \to Qc$, $\exists x(Px \to Qc)$

   domain: 0, 1
   P: 0
   Q:
   c: 0

   This interpretation obviously makes '$\exists xPx \to Qc$' false. It also makes '$\exists x(Px \to Qc)$' true: 'Px' is false of 1, and so '$Px \to Qc$' is vacuously true of 1.
5. $\forall x(Px \to \neg Qx)$, $\exists x(Px \land \neg Qx)$

   domain: 0
   P:
   Q:

   This interpretation makes '$\forall x(Px \to \neg Qx)$' vacuously true, and '$\exists x(Px \land \neg Qx)$' false.
6. $\exists x(Px \land Qx)$, $\exists x(Px \to Qx)$

   domain: 0
   P:

Q:

This interpretation obviously makes '$\exists x(Px \wedge Qx)$' false. It also makes '$\exists x(Px \rightarrow Qx)$' true: '$Px$' is false of 0, and so '$Px \rightarrow Qx$' is vacuously true of 0.

7. $\forall x(Px \rightarrow Qx)$, $\forall x(Px \wedge Qx)$

   domain: 0
   P:
   Q:

   This interpretation obviously makes '$\forall x(Px \wedge Qx)$' false. It also makes '$\forall x(Px \rightarrow Qx)$' true: '$Px$' is false of 0, so '$Px \rightarrow Qx$' is vacuously true of 0, and 0 is the only object in the domain.

8. $\forall x \exists y Rxy$, $\exists x \forall y Rxy$

   domain: 0, 1
   R: $<0,1>$, $<1,0>$

   This interpretation makes '$\forall x \exists y Rxy$' true and '$\exists x \forall y Rxy$' false.

9. $\forall x \exists y Rxy$, $\forall x \exists y Ryx$

   domain: 0, 1
   R: $<0,1>$, $<1,1>$

   This interpretation makes '$\forall x \exists y Rxy$' true and '$\forall x \exists y Ryx$' false.

C. Show that the following sentences are jointly consistent in FOL:
To show that a collection of sentences are jointly consistent in FOL, we just need to present an interpretation which makes all of those sentences true.

1. $Ma, \neg Na, Pa, \neg Qa$

   domain: 0
   M: 0
   N:
   P: 0
   Q:
   a: 0

2. $Lee, Leg, \neg Lge, \neg Lgg$

   domain: 0, 1
   L: $<0,0>$, $<0,1>$
   e: 0
   g: 1

3. $\neg(Ma \wedge \exists x Ax), Ma \vee Fa, \forall x(Fx \rightarrow Ax)$

   domain: 0
   M:
   F: 0
   A: 0
   a: 0

4. $Ma \vee Mb, Ma \rightarrow \forall x \neg Mx$

   domain: 0,1
   M: 1
   a: 0
   b: 1

   This sentence obviously makes '$Ma \vee Mb$' true. It also makes '$Ma \rightarrow \forall x \neg Mx$' vacuously true, since it makes 'Ma' false.

5. $\forall y Gy, \forall x(Gx \rightarrow Hx), \exists y \neg Iy$

   domain: 0
   G: 0
   H: 0
   I:

6. $\exists x(Bx \vee Ax), \forall x \neg Cx, \forall x\big[(Ax \wedge Bx) \rightarrow Cx\big]$

   domain: 0
   A: 0
   B:
   C:

7. $\exists x Xx, \exists x Yx, \forall x(Xx \leftrightarrow \neg Yx)$

   domain: 0, 1
   X: 0
   Y: 1

   This interpretation obviously makes '$\exists x Xx$' and '$\exists x Yx$' true. It also makes '$\forall x(Xx \leftrightarrow \neg Yx)$' true: '$Xx \leftrightarrow \neg Yx$' is true of 0, since '$Xx$' is true of 0 but '$Yx$' isn't, and '$Xx \leftrightarrow \neg Yx$' is true of 1, since '$Yx$' is true of 1 and '$Xx$' isn't.

8. $\forall x(Px \vee Qx), \exists x \neg(Qx \wedge Px)$

   domain: 0
   P: 0
   Q:

9. $\exists z(Nz \wedge Ozz), \forall x \forall y(Oxy \rightarrow Oyx)$

   domain: 0
   N: 0
   O: <0,0>

10. $\neg \exists x \forall y Rxy, \forall x \exists y Rxy$

    domain: 0, 1
    R: <0,1>, <1,0>

11. $\neg Raa$, $\forall x(x = a \lor Rxa)$

    domain: 0
    R:
    a: 0

12. $\forall x \forall y \forall z(x = y \lor y = z \lor x = z)$, $\exists x \exists y\, \neg x = y$

    domain: 0, 1

    This interpretation makes '$\forall x \forall y \forall z(x = y \lor y = z \lor x = z)$' true: this sentence is true just in case there are no more than 2 objects in the domain. It also makes '$\exists x \exists y\, \neg x = y$' true: this sentence is true just in case there are at least 2 objects in the domain.

13. $\exists x \exists y(Zx \land Zy \land x = y)$, $\neg Zd$, $d = e$

    domain: 0, 1
    Z: 0
    d: 1
    e: 1

D. Show that the following arguments are invalid in FOL:
To show that an FOL argument is invalid in FOL, we need to present an interpretation which makes all of the premises true, and makes the conclusion false.

1. $\forall x(Ax \rightarrow Bx) \therefore \exists x Bx$

    domain: 0
    A:
    B:

2. $\forall x(Rx \rightarrow Dx), \forall x(Rx \rightarrow Fx) \therefore \exists x(Dx \land Fx)$

    domain: 0
    R:
    D:
    F:

3. $\exists x(Px \rightarrow Qx) \therefore \exists x Px$

    domain: 0
    P:
    Q:

4. $Na \land Nb \land Nc \therefore \forall x Nx$

    domain: 0, 1
    N: 0
    a: 0
    b: 0
    c: 0

5. $Rde, \exists x Rxd \therefore Red$

    domain: 0, 1, 2
    R: <0,1>, <2,0>
    d: 0
    e: 1

6. $\exists x(Ex \wedge Fx), \exists xFx \rightarrow \exists xGx \therefore \exists x(Ex \wedge Gx)$

   domain: 0, 1
   E: 0
   F: 0
   G: 1

7. $\forall xOxc, \forall xOcx \therefore \forall xOxx$

   domain: 0, 1
   O: <0,1>, <1,1>, <1,0>
   c: 1

8. $\exists x(Jx \wedge Kx), \exists x\neg Kx, \exists x\neg Jx \therefore \exists x(\neg Jx \wedge \neg Kx)$

   domain: 0, 1, 2
   J: 0, 1
   K: 0, 2

9. $Lab \rightarrow \forall xLxb, \exists xLxb \therefore Lbb$

   domain: 0, 1, 2
   L: <2,1>
   a: 0
   b: 1

10. $\forall x(Dx \rightarrow \exists yTyx) \therefore \exists y\exists z\, \neg y = z$

   domain: 0
   D: 0
   T: <0,0>

   We could have made the premise true just by giving 'D' and 'T' empty extensions. However, the above interpretation reminds us of an important point: '$\exists yTyx$' should **not** be read as saying: there is some y *other than* x such that Tyx.

Basic rules for FOL 30

A. The following two 'proofs' are *incorrect*. Explain why both are incorrect. Also, provide interpretations which show that the corresponding arguments are invalid in FOL:

1	$\forall x Rxx$	
2	Raa	$\forall$E 1
3	$\forall y Ray$	$\forall$I 2
4	$\forall x \forall y Rxy$	$\forall$I 3

When using $\forall$I, you must replace *all* names with the new variable. So line 3 is bogus. As a counterinterpretation, consider the following:

1	$\forall x \exists y Rxy$	
2	$\exists y Ray$	$\forall$E 1
3	$\quad Raa$	
4	$\quad \exists x Rxx$	$\exists$I 3
5	$\exists x Rxx$	$\exists$E 2, 3–4

The instantiating constant, 'a', occurs in the line (line 2) to which $\exists$E is to be applied on line 5. So the use of $\exists$E on line 5 is bogus. As a counterinterpretation, consider the following:

$$1 \longleftrightarrow 2$$

B. The following three proofs are missing their citations (rule and line numbers). Add them, to turn them into bona fide proofs.

1	$\forall x \exists y (Rxy \lor Ryx)$	
2	$\forall x \neg Rmx$	
3	$\exists y (Rmy \lor Rym)$	$\forall$E 1
4	$\quad Rma \lor Ram$	
5	$\quad \neg Rma$	$\forall$E 2
6	$\quad Ram$	DS 4, 5
7	$\quad \exists x Rxm$	$\exists$I 6
8	$\exists x Rxm$	$\exists$E 3, 4–7

1	$\forall x (\exists y Lxy \to \forall z Lzx)$	
2	Lab	
3	$\exists y Lay \to \forall z Lza$	$\forall$E 1
4	$\exists y Lay$	$\exists$I 2
5	$\forall z Lza$	$\to$E 3, 4
6	Lca	$\forall$E 5
7	$\exists y Lcy \to \forall z Lzc$	$\forall$E 1
8	$\exists y Lcy$	$\exists$I 6
9	$\forall z Lzc$	$\to$E 7, 8
10	Lcc	$\forall$E 9
11	$\forall x Lxx$	$\forall$I 10

1	$\forall x(Jx \to Kx)$	
2	$\exists x \forall y Lxy$	
3	$\forall x Jx$	
4	$\quad \forall y Lay$	
5	$\quad Laa$	$\forall$E 4
6	$\quad Ja$	$\forall$E 3
7	$\quad Ja \to Ka$	$\forall$E 1
8	$\quad Ka$	$\to$E 7, 6
9	$\quad Ka \wedge Laa$	$\wedge$I 8, 5
10	$\quad \exists x(Kx \wedge Lxx)$	$\exists$I 9
11	$\exists x(Kx \wedge Lxx)$	$\exists$E 2, 4–10

C. In §20 problem part A, we considered fifteen syllogistic figures of Aristotelian logic. Provide proofs for each of the argument forms. NB: You will find it *much* easier if you symbolise (for example) 'No F is G' as '$\forall x(Fx \to \neg Gx)$'.
I shall prove the four Figure I syllogisms; the rest are *extremely* similar.

Barbara

1	$\forall x(Gx \to Fx)$	
2	$\forall x(Hx \to Gx)$	
3	$Ga \to Fa$	$\forall$E 1
4	$Ha \to Ga$	$\forall$E 2
5	$\quad Ha$	
6	$\quad Ga$	$\to$E 4, 5
7	$\quad Fa$	$\to$E 3, 6
8	$Ha \to Fa$	$\to$I 5–7
9	$\forall x(Hx \to Fx)$	$\forall$I 8

Celerant is exactly as Barbara, replacing 'F' with '$\neg F$' throughout.

Ferio

1	$\forall x(Gx \to \neg Fx)$	
2	$\exists x(Hx \wedge Gx)$	
3	$\quad Ha \wedge Ga$	
4	$\quad Ha$	$\wedge$E 3
5	$\quad Ga$	$\wedge$E 3
6	$\quad Ga \to \neg Fa$	$\forall$E 1
7	$\quad \neg Fa$	$\to$E 6, 5
8	$\quad Ha \wedge \neg Fa$	$\wedge$I 4, 7
9	$\quad \exists x(Hx \wedge \neg Fx)$	$\exists$I 8
10	$\exists x(Hx \wedge \neg Fx)$	$\exists$E 2, 3–9

Darii is exactly as Ferio, replacing '$\neg F$' with 'F' throughout.

D. Aristotle and his successors identified other syllogistic forms which depended upon 'existential import'. Symbolise each of the following argument forms in FOL and offer proofs.

- **Barbari.** Something is H. All G are F. All H are G. So: Some H is F
  $\exists x Hx, \forall x(Gx \rightarrow Fx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge Fx)$

1	$\exists x Hx$	
2	$\forall x(Gx \rightarrow Fx)$	
3	$\forall x(Hx \rightarrow Gx)$	
4	$\quad Ha$	
5	$\quad Ha \rightarrow Ga$	$\forall$E 3
6	$\quad Ga$	$\rightarrow$E 5, 4
7	$\quad Ga \rightarrow Fa$	$\forall$E 2
8	$\quad Fa$	$\rightarrow$E 7, 6
9	$\quad Ha \wedge Fa$	$\wedge$I 4, 8
10	$\quad \exists x(Hx \wedge Fx)$	$\exists$I 9
11	$\exists x(Hx \wedge Fx)$	$\exists$E 1, 4–10

- **Celaront.** Something is H. No G are F. All H are G. So: Some H is not F
  $\exists x Hx, \forall x(Gx \rightarrow \neg Fx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge \neg Fx)$
  Proof is exactly as for Barbari, replacing 'F' with '$\neg F$' throughout.
- **Cesaro.** Something is H. No F are G. All H are G. So: Some H is not F.
  $\exists x Hx, \forall x(Fx \rightarrow \neg Gx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge \neg Fx)$

1	$\exists x Hx$	
2	$\forall x(Fx \rightarrow \neg Gx)$	
3	$\forall x(Hx \rightarrow Gx)$	
4	$\quad Ha$	
5	$\quad Ha \rightarrow Ga$	$\forall$E 3
6	$\quad Ga$	$\rightarrow$E 5, 4
7	$\quad Fa \rightarrow \neg Ga$	$\forall$E 2
8	$\quad\quad Fa$	
9	$\quad\quad \neg Ga$	$\rightarrow$E 7, 8
10	$\quad\quad \bot$	$\bot$I 6, 9
11	$\quad \neg Fa$	$\neg$I 8–10
12	$\quad Ha \wedge \neg Fa$	$\wedge$I 4, 11
13	$\quad \exists x(Hx \wedge \neg Fx)$	$\exists$I 12
14	$\exists x(Hx \wedge \neg Fx)$	$\exists$E 1, 4–13

- **Camestros.** Something is H. All F are G. No H are G. So: Some H is not F.
  $\exists x Hx, \forall x(Fx \rightarrow Gx), \forall x(Hx \rightarrow \neg Gx) \therefore \exists x(Hx \wedge \neg Fx)$

1	$\exists x Hx$	
2	$\forall x(Fx \rightarrow Gx)$	
3	$\forall x(Hx \rightarrow \neg Gx)$	
4	$\quad Ha$	
5	$\quad Ha \rightarrow \neg Ga$	$\forall$E 3
6	$\quad \neg Ga$	$\rightarrow$E 5, 4
7	$\quad Fa \rightarrow Ga$	$\forall$E 2
8	$\quad \neg Fa$	MT 7, 6
9	$\quad Ha \wedge \neg Fa$	$\wedge$I 4, 8
10	$\quad \exists x(Hx \wedge \neg Fx)$	$\exists$I 9
11	$\exists x(Hx \wedge \neg Fx)$	$\exists$E 1, 4–10

- **Felapton.** Something is G. No G are F. All G are H. So: Some H is not F.
  $\exists x Gx, \forall x(Gx \rightarrow \neg Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge \neg Fx)$

1	$\exists x Gx$	
2	$\forall x(Gx \rightarrow \neg Fx)$	
3	$\forall x(Gx \rightarrow Hx)$	
4	$\quad Ga$	
5	$\quad Ga \rightarrow Ha$	$\forall$E 3
6	$\quad Ha$	$\rightarrow$E 5, 4
7	$\quad Ga \rightarrow \neg Fa$	$\forall$E 2
8	$\quad \neg Fa$	$\rightarrow$E 7, 4
9	$\quad Ha \wedge \neg Fa$	$\wedge$I 6, 8
10	$\quad \exists x(Hx \wedge \neg Fx)$	$\exists$I 9
11	$\exists x(Hx \wedge \neg Fx)$	$\exists$E 1, 4–10

- **Darapti.** Something is G. All G are F. All G are H. So: Some H is F.
  $\exists x Gx, \forall x(Gx \rightarrow Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge Fx)$
  Proof is exactly as for Felapton, replacing '$\neg F$' with 'F' throughout.

- **Calemos.** Something is H. All F are G. No G are H. So: Some H is not F.
  $\exists x Hx, \forall x(Fx \rightarrow Gx), \forall x(Gx \rightarrow \neg Hx) \therefore \exists x(Hx \land \neg Fx)$

1	$\exists x Hx$	
2	$\forall x(Fx \rightarrow Gx)$	
3	$\forall x(Gx \rightarrow \neg Hx)$	
4	$\quad Ha$	
5	$\quad Ga \rightarrow \neg Ha$	$\forall$E 3
6	$\quad\quad Ga$	
7	$\quad\quad \neg Ha$	$\rightarrow$E 5, 6
8	$\quad\quad \bot$	$\bot$I 4, 7
9	$\quad \neg Ga$	$\neg$I 6–8
10	$\quad Fa \rightarrow Ga$	$\forall$E 2
11	$\quad \neg Fa$	MT 10, 9
12	$\quad Ha \land \neg Fa$	$\land$I 4, 11
13	$\quad \exists x(Hx \land \neg Fx)$	$\exists$I 12
14	$\exists x(Hx \land \neg Fx)$	$\exists$E 1, 4–13

- **Fesapo.** Something is G. No F is G. All G are H. So: Some H is not F.
  $\exists x Gx, \forall x(Fx \rightarrow \neg Gx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \land \neg Fx)$

1	$\exists x Gx$	
2	$\forall x(Fx \rightarrow \neg Gx)$	
3	$\forall x(Gx \rightarrow Hx)$	
4	$\quad Ga$	
5	$\quad Ga \rightarrow Ha$	$\forall$E 3
6	$\quad Ha$	$\rightarrow$E 5, 4
7	$\quad Fa \rightarrow \neg Ga$	$\forall$E 2
8	$\quad\quad Fa$	
9	$\quad\quad \neg Ga$	$\rightarrow$E 7, 8
10	$\quad\quad \bot$	$\bot$I 4, 9
11	$\quad \neg Fa$	$\neg$I 8–10
12	$\quad Ha \land \neg Fa$	$\land$I 6, 11
13	$\quad \exists x(Hx \land \neg Fx)$	$\exists$I 12
14	$\exists x(Hx \land \neg Fx)$	$\exists$E 1, 4–13

- **Bamalip.** Something is F. All F are G. All G are H. So: Some H are F.
  $\exists x Fx, \forall x(Fx \to Gx), \forall x(Gx \to Hx) \therefore \exists x(Hx \land Fx)$

1	$\exists x Fx$	
2	$\forall x(Fx \to Gx)$	
3	$\forall x(Gx \to Hx)$	
4	$\mid\ Fa$	
5	$\mid\ Fa \to Ga$	$\forall$E 2
6	$\mid\ Ga$	$\to$E 5, 4
7	$\mid\ Ga \to Ha$	$\forall$E 3
8	$\mid\ Ha$	$\to$E 7, 6
9	$\mid\ Ha \land Fa$	$\land$I 8, 4
10	$\mid\ \exists x(Hx \land Fx)$	$\exists$I 9
11	$\exists x(Hx \land Fx)$	$\exists$E 1, 4–10

E. Provide a proof of each claim.

1. $\vdash \forall x Fx \lor \neg\forall x Fx$

1	$\mid\ \forall x Fx$	
2	$\mid\ \forall x Fx \lor \neg\forall x Fx$	$\lor$I 1
3	$\mid\ \neg\forall x Fx$	
4	$\mid\ \forall x Fx \lor \neg\forall x Fx$	$\lor$I 3
5	$\forall x Fx \lor \neg\forall x Fx$	TND 1–2, 3–4

2. $\vdash \forall z(Pz \lor \neg Pz)$

1	$\mid\ Pa$	
2	$\mid\ Pa \lor \neg Pa$	$\lor$I 1
3	$\mid\ \neg Pa$	
4	$\mid\ Pa \lor \neg Pa$	$\lor$I 3
5	$Pa \lor \neg Pa$	TND 1–2, 3–4
6	$\forall z(Pz \lor \neg Pz)$	$\forall$I 5

3. $\forall x(Ax \to Bx), \exists x Ax \vdash \exists x Bx$

1	$\forall x(Ax \to Bx)$	
2	$\exists x Ax$	
3	$\mid\ Aa$	
4	$\mid\ Aa \to Ba$	$\forall$E 1
5	$\mid\ Ba$	$\to$E 4, 3
6	$\mid\ \exists x Bx$	$\exists$I 5
7	$\exists x Bx$	$\exists$E 2, 3–6

4. $\forall x(Mx \leftrightarrow Nx), Ma \wedge \exists x Rxa \vdash \exists x Nx$

1	$\forall x(Mx \leftrightarrow Nx)$	
2	$Ma \wedge \exists x Rxa$	
3	Ma	$\wedge$E 2
4	$Ma \leftrightarrow Na$	$\forall$E 1
5	Na	$\leftrightarrow$E 4, 3
6	$\exists x Nx$	$\exists$I 5

5. $\forall x \forall y Gxy \vdash \exists x Gxx$

1	$\forall x \forall y Gxy$	
2	$\forall y Gay$	$\forall$E 1
3	Gaa	$\forall$E 2
4	$\exists x Gxx$	$\exists$I 3

6. $\vdash \forall x Rxx \to \exists x \exists y Rxy$

1	$\quad \forall x Rxx$	
2	$\quad Raa$	$\forall$E 1
3	$\quad \exists y Ray$	$\exists$I 2
4	$\quad \exists x \exists y Rxy$	$\exists$I 3
5	$\forall x Rxx \to \exists x \exists y Rxy$	$\to$I 1–4

7. $\vdash \forall y \exists x(Qy \to Qx)$

1	$\quad Qa$	
2	$\quad Qa$	R 1
3	$Qa \to Qa$	$\to$I 1–2
4	$\exists x(Qa \to Qx)$	$\exists$I 3
5	$\forall y \exists x(Qy \to Qx)$	$\forall$I 4

8. $Na \to \forall x(Mx \leftrightarrow Ma), Ma, \neg Mb \vdash \neg Na$

1	$Na \to \forall x(Mx \leftrightarrow Ma)$	
2	Ma	
3	$\neg Mb$	
4	$\quad Na$	
5	$\quad \forall x(Mx \leftrightarrow Ma)$	$\to$E 1, 4
6	$\quad Mb \leftrightarrow Ma$	$\forall$E 5
7	$\quad Mb$	$\leftrightarrow$E 6, 2
8	$\quad \bot$	$\bot$I 7, 3
9	$\neg Na$	$\neg$I 4–8

9. $\forall x \forall y(Gxy \to Gyx) \vdash \forall x \forall y(Gxy \leftrightarrow Gyx)$

1	$\forall x \forall y(Gxy \to Gyx)$	
2	$\quad Gab$	
3	$\quad \forall y(Gay \to Gya)$	$\forall$E 1
4	$\quad Gab \to Gba$	$\forall$E 3
5	$\quad Gba$	$\to$E 4, 2
6	$\quad Gba$	
7	$\quad \forall y(Gby \to Gyb)$	$\forall$E 1
8	$\quad Gba \to Gab$	$\forall$E 7
9	$\quad Gab$	$\to$E 8, 6
10	$Gab \leftrightarrow Gba$	$\leftrightarrow$I 2–5, 6–9
11	$\forall y(Gay \leftrightarrow Gya)$	$\forall$I 10
12	$\forall x \forall y(Gxy \leftrightarrow Gyx)$	$\forall$I 11

10. $\forall x(\neg Mx \lor Ljx), \forall x(Bx \to Ljx), \forall x(Mx \lor Bx) \vdash \forall x Ljx$

1	$\forall x(\neg Mx \lor Ljx)$	
2	$\forall x(Bx \to Ljx)$	
3	$\forall x(Mx \lor Bx)$	
4	$\neg Ma \lor Lja$	$\forall$E 1
5	$Ba \to Lja$	$\forall$E 2
6	$Ma \lor Ba$	$\forall$E 3
7	$\quad \neg Ma$	
8	$\quad Ba$	DS 6, 7
9	$\quad Lja$	$\to$E 5, 8
10	$\quad Lja$	
11	$\quad Lja$	R 10
12	Lja	$\lor$E 4, 7–9, 10–11
13	$\forall x Ljx$	$\forall$I 12

F. Write a symbolisation key for the following argument, symbolise it, and prove it:

> There is someone who likes everyone who likes everyone that she likes. Therefore, there is someone who likes herself.

Symbolisation key:

domain: all people
Lxy: _____$_x$ likes _____$_y$

$\exists x \forall y(\forall z(Lxz \to Lyz) \to Lxy) \therefore \exists x Lxx$

1	$\exists x\forall y(\forall z(Lxz \to Lyz) \to Lxy)$	
2	$\quad \forall y(\forall z(Laz \to Lyz) \to Lay)$	
3	$\quad \forall z(Laz \to Laz) \to Laa$	$\forall$E 2
4	$\quad\quad Lac$	
5	$\quad\quad Lac$	R 4
6	$\quad Lac \to Lac$	$\to$I 4–5
7	$\quad \forall z(Laz \to Laz)$	$\forall$I 6
8	$\quad Laa$	$\to$E 3, 7
9	$\quad \exists xLxx$	$\exists$I 8
10	$\exists xLxx$	$\exists$E 1, 2—9

G. For each of the following pairs of sentences: If they are provably equivalent, give proofs to show this. If they are not, construct an interpretation to show that they are not equivalent in FOL.

1. $\forall xPx \to Qc, \forall x(Px \to Qc)$ Not equivalent in FOL
   Counter-interpretation: let the domain be the numbers 1 and 2. Let 'c' name 1. Let 'Px' be true of and only of 1. Let 'Qx' be true of, and only of, 2.
2. $\forall x\forall y\forall zBxyz, \forall xBxxx$ Not equivalent in FOL
   Counter-interpretation: let the domain be the numbers 1 and 2. Let '$Bxyz$' be true of, and only of, <1,1,1> and <2,2,2>.
3. $\forall x\forall yDxy, \forall y\forall xDxy$ Provably equivalent

1	$\forall x\forall yDxy$	
2	$\forall yDay$	$\forall$E 1
3	Dab	$\forall$E 2
4	$\forall xDxb$	$\forall$I 3
5	$\forall y\forall xDxy$	$\forall$I 4

1	$\forall y\forall xDxy$	
2	$\forall xDxa$	$\forall$E 1
3	Dba	$\forall$E 2
4	$\forall yDby$	$\forall$I 3
5	$\forall x\forall yDxy$	$\forall$I 4

4. $\exists x\forall yDxy, \forall y\exists xDxy$ Not equivalent in FOL
   Counter-interpretation: let the domain be the numbers 1 and 2. Let 'Dxy' hold of and only of <1,2> and <2,1>. This is depicted thus:

$$1 \longleftrightarrow 2$$

5. $\forall x(Rca \leftrightarrow Rxa), Rca \leftrightarrow \forall xRxa$ Not equivalent in FOL
   Counter-interpretation, consider the following diagram, allowing 'a' to name 1 and 'c' to name 2:

H. For each of the following arguments: If it is valid in FOL, give a proof. If it is invalid in FOL, construct an interpretation to show that it is invalid in FOL.

1. $\exists y \forall x Rxy \therefore \forall x \exists y Rxy$ — Valid in FOL

1	$\exists y \forall x Rxy$	
2	$\vert\ \forall x Rxa$	
3	$\vert\ Rba$	$\forall$E 2
4	$\vert\ \exists y Rby$	$\exists$I 3
5	$\exists y Rby$	$\exists$E 1, 2–4
6	$\forall x \exists y Rxy$	$\forall$I 5

2. $\exists x(Px \wedge \neg Qx) \therefore \forall x(Px \rightarrow \neg Qx)$ — Invalid in FOL
   Counter interpretation: let the domain be the numbers 1 and 2. Let 'Px' be true of everything in the domain. Let 'Qx' be true of, and only of, 2.
3. $\forall x(Sx \rightarrow Ta), Sd \therefore Ta$ — Valid in FOL

1	$\forall x(Sx \rightarrow Ta)$	
2	Sd	
3	$Sd \rightarrow Ta$	$\forall$E 1
4	Ta	$\rightarrow$E 3, 2

4. $\forall x(Ax \rightarrow Bx), \forall x(Bx \rightarrow Cx) \therefore \forall x(Ax \rightarrow Cx)$ — Valid in FOL

1	$\forall x(Ax \rightarrow Bx)$	
2	$\forall x(Bx \rightarrow Cx)$	
3	$Aa \rightarrow Ba$	$\forall$E 1
4	$Ba \rightarrow Ca$	$\forall$E 2
5	$\vert\ Aa$	
6	$\vert\ Ba$	$\rightarrow$E 3, 5
7	$\vert\ Ca$	$\rightarrow$E 4, 6
8	$Aa \rightarrow Ca$	$\rightarrow$I 5–7
9	$\forall x(Ax \rightarrow Cx)$	$\forall$I 8

5. $\exists x(Dx \vee Ex), \forall x(Dx \rightarrow Fx) \therefore \exists x(Dx \wedge Fx)$ — Invalid in FOL
   Counter-interpretation: let the domain be the number 1 . Let 'Dx' hold of nothing. Let both 'Ex' and 'Fx' hold of everything.
6. $\forall x \forall y(Rxy \vee Ryx) \therefore Rjj$ — Valid in FOL

1	$\forall x \forall y(Rxy \lor Ryx)$	
2	$\forall y(Rjy \lor Ryj)$	$\forall$E 1
3	$Rjj \lor Rjj$	$\forall$E 2
4	$\quad Rjj$	
5	$\quad Rjj$	R 4
6	$\quad Rjj$	
7	$\quad Rjj$	R 6
8	Rjj	$\lor$E 3, 4–5, 6–7

7. $\exists x \exists y(Rxy \lor Ryx) \therefore Rjj$ Invalid in FOL
   Counter-interpretation: consider the following diagram, allowing 'j' to name 2.

8. $\forall x Px \to \forall x Qx, \exists x \neg Px \therefore \exists x \neg Qx$ Invalid in FOL
   Counter-interpretation: let the domain be the number 1. Let 'Px' be true of nothing. Let 'Qx' be true of everything.

Conversion of quantifiers 31

A. Offer proofs which justify the addition of the third and fourth CQ rules as derived rules.

Justification for the third rule:

$$
\begin{array}{lll}
1 & \mid \neg\exists x Ax & \\
\hline
2 & \mid \quad \mid Aa & \\
3 & \mid \quad \mid \exists x Ax & \exists\text{I } 2 \\
4 & \mid \quad \mid \bot & \bot\text{I } 3, 1 \\
5 & \mid \neg Aa & \neg\text{I } 2\text{–}4 \\
6 & \mid \forall x \neg Ax & \forall\text{I } 5
\end{array}
$$

Justification for the fourth rule:

$$
\begin{array}{lll}
1 & \mid \forall x \neg Ax & \\
\hline
2 & \mid \quad \mid \exists x Ax & \\
3 & \mid \quad \mid \quad \mid Aa & \\
4 & \mid \quad \mid \quad \mid \neg Aa & \forall\text{E } 1 \\
5 & \mid \quad \mid \quad \mid \bot & \bot\text{I } 3, 4 \\
6 & \mid \quad \mid \bot & \exists\text{E } 2, 3\text{–}5 \\
7 & \mid \neg\exists x Ax & \neg\text{I } 2\text{–}6
\end{array}
$$

B. Show that the following are jointly contrary:

1. $Sa \to Tm, Tm \to Sa, Tm \land \neg Sa$

$$
\begin{array}{lll}
1 & \mid Sa \to Tm & \\
2 & \mid Tm \to Sa & \\
3 & \mid Tm \land \neg Sa & \\
\hline
4 & \mid Tm & \land\text{E } 3 \\
5 & \mid \neg Sa & \land\text{E } 3 \\
6 & \mid Sa & \to\text{E } 2, 4 \\
7 & \mid \bot & \bot\text{I } 5, 6
\end{array}
$$

2. $\neg\exists x Rxa, \forall x \forall y Ryx$

1	$\neg\exists x Rxa$	
2	$\forall x \forall y Ryx$	
3	$\forall x \neg Rxa$	CQ 1
4	$\neg Rba$	∀E 3
5	$\forall y Rya$	∀E 2
6	Rba	∀E 5
7	$\bot$	⊥I 6, 4

3. $\neg\exists x \exists y Lxy, Laa$

1	$\neg\exists x \exists y Lxy$	
2	Laa	
3	$\forall x \neg\exists y Lxy$	CQ 1
4	$\neg\exists y Lay$	∀E 3
5	$\forall y \neg Lay$	CQ 4
6	$\neg Laa$	∀E 5
7	$\bot$	⊥I 2, 6

4. $\forall x(Px \to Qx), \forall z(Pz \to Rz), \forall y Py, \neg Qa \wedge \neg Rb$

1	$\forall x(Px \to Qx)$	
2	$\forall z(Pz \to Rz)$	
3	$\forall y Py$	
4	$\neg Qa \wedge \neg Rb$	
5	$\neg Qa$	∧E 4
6	$Pa \to Qa$	∀E 1
7	$\neg Pa$	MT 6, 5
8	Pa	∀E 3
9	$\bot$	⊥I 8, 7

C. Show that each pair of sentences is provably equivalent:

1. $\forall x(Ax \to \neg Bx), \neg\exists x(Ax \wedge Bx)$

1	$\forall x(Ax \to \neg Bx)$	
2	$\quad \exists x(Ax \wedge Bx)$	
3	$\quad\quad Aa \wedge Ba$	
4	$\quad\quad Aa$	∧E 3
5	$\quad\quad Ba$	∧E 3
6	$\quad\quad Aa \to \neg Ba$	∀E 1
7	$\quad\quad \neg Ba$	→E 6, 4
8	$\quad\quad \bot$	⊥I 5, 7
9	$\quad \bot$	∃E 2, 3–8
10	$\neg\exists x(Ax \wedge Bx)$	¬I 2–9

1	$\neg\exists x(Ax \wedge Bx)$	
2	$\forall x\neg(Ax \wedge Bx)$	CQ 1
3	$\neg(Aa \wedge Ba)$	$\forall$E 2
4	$\quad\mid Aa$	
5	$\quad\mid\quad\mid Ba$	
6	$\quad\mid\quad\mid Aa \wedge Ba$	$\wedge$I 4, 5
7	$\quad\mid\quad\mid \bot$	$\bot$I 6, 3
8	$\quad\mid \neg Ba$	$\neg$I 5–7
9	$Aa \to \neg Ba$	$\to$I 4–8
10	$\forall x(Ax \to \neg Bx)$	$\forall$I 9

2. $\forall x(\neg Ax \to Bd), \forall xAx \vee Bd$

1	$\forall x(\neg Ax \to Bd)$	
2	$\neg Aa \to Bd$	$\forall$E 1
3	$\quad\mid Bd$	
4	$\quad\mid \forall xAx \vee Bd$	$\vee$I 6
5	$\quad\mid \neg Bd$	
6	$\quad\mid \neg\neg Aa$	MT 2, 5
7	$\quad\mid Aa$	DNE 6
8	$\quad\mid \forall xAx$	$\forall$E 7
9	$\quad\mid \forall xAx \vee Bd$	$\vee$I 8
10	$\forall xAx \vee Bd$	TND 3–4, 5–9

1	$\forall xAx \vee Bd$	
2	$\quad\mid \neg Aa$	
3	$\quad\mid\quad\mid \forall xAx$	
4	$\quad\mid\quad\mid Aa$	$\forall$E 3
5	$\quad\mid\quad\mid \bot$	$\bot$I 4, 2
6	$\quad\mid \neg\forall xAx$	$\neg$I 3–5
7	$\quad\mid Bd$	DS 1, 6
8	$\neg Aa \to Bd$	$\to$I 2–7
9	$\forall x(Ax \to Bd)$	$\forall$I 8

D. In §20, we considered what happens when we move quantifiers 'across' various logical operators. Show that each pair of sentences is provably equivalent:

1. $\forall x(Fx \wedge Ga), \forall xFx \wedge Ga$

1	$\forall x(Fx \wedge Ga)$	
2	$Fb \wedge Ga$	$\forall$E 1
3	Fb	$\wedge$E 2
4	Ga	$\wedge$E 6
5	$\forall xFx$	$\forall$I 3
6	$\forall xFx \wedge Ga$	$\wedge$I 5, 4

1	$\forall xFx \wedge Ga$	
2	$\forall xFx$	$\wedge$E 1
3	Ga	$\wedge$E 1
4	Fb	$\forall$E 2
5	$Fb \wedge Ga$	$\wedge$I 4, 3
6	$\forall x(Fx \wedge Ga)$	$\forall$I 5

2. $\exists x(Fx \vee Ga), \exists xFx \vee Ga$

1	$\exists x(Fx \vee Ga)$	
2	$Fb \vee Ga$	
3	Fb	
4	$\exists xFx$	$\exists$I 3
5	$\exists xFx \vee Ga$	$\vee$I 4
6	Ga	
7	$\exists xFx \vee Ga$	$\vee$I 6
8	$\exists xFx \vee Ga$	$\vee$E 2, 3–5, 6–7
9	$\exists xFx \vee Ga$	$\exists$E 1, 2–8

1	$\exists xFx \vee Ga$	
2	$\exists xFx$	
3	Fb	
4	$Fb \vee Ga$	$\vee$I 3
5	$\exists x(Fx \vee Ga)$	$\exists$I 4
6	$\exists x(Fx \vee Ga)$	$\exists$E 2, 3–5
7	Ga	
8	$Fb \vee Ga$	$\vee$I 7
9	$\exists x(Fx \vee Ga)$	$\exists$I 8
10	$\exists x(Fx \vee Ga)$	$\vee$E 1, 2–6, 7–9

3. $\forall x(Ga \to Fx), Ga \to \forall xFx$

1	$\forall x(Ga \to Fx)$	
2	$Ga \to Fb$	$\forall$E 1
3	Ga	
4	Fb	$\to$E 2, 3
5	$\forall xFx$	$\forall$I 4
6	$Ga \to \forall xFx$	$\to$I 3–5

1	$Ga \to \forall xFx$	
2	Ga	
3	$\forall xFx$	$\to$E 1, 2
4	Fb	$\forall$E 3
5	$Ga \to Fb$	$\to$I 2–4
6	$\forall x(Ga \to Fx)$	$\forall$I 5

4. $\forall x(Fx \to Ga), \exists xFx \to Ga$

1	$\forall x(Fx \to Ga)$	
2	$\exists xFx$	
3	Fb	
4	$Fb \to Ga$	$\forall$E 1
5	Ga	$\to$E 4, 3
6	Ga	$\exists$E 2, 3–5
7	$\exists xFx \to Ga$	$\to$I 2–6

1	$\exists xFx \to Ga$	
2	Fb	
3	$\exists xFx$	$\exists$I 2
4	Ga	$\to$E 1, 3
5	$Fb \to Ga$	$\to$I 2–4
6	$\forall x(Fx \to Ga)$	$\forall$I 5

5. $\exists x(Ga \to Fx), Ga \to \exists xFx$

1	$\exists x(Ga \to Fx)$	
2	$\quad Ga$	
3	$\quad\quad Ga \to Fb$	
4	$\quad\quad Fb$	$\to$E 3, 2
5	$\quad\quad \exists xFx$	$\exists$I 4
6	$\quad \exists xFx$	$\exists$E 1, 3–5
7	$Ga \to \exists xFx$	$\to$I 2–6

1	$Ga \to \exists xFx$	
2	$\quad Ga$	
3	$\quad \exists xFx$	
4	$\quad\quad Fb$	
5	$\quad\quad\quad Ga$	
6	$\quad\quad\quad Fb$	R 4
7	$\quad\quad Ga \to Fb$	$\to$I 5–6
8	$\quad\quad \exists x(Ga \to Fx)$	$\exists$I 7
9	$\quad \exists x(Ga \to Fx)$	$\exists$E 3, 4–8
10	$\quad \neg Ga$	
11	$\quad\quad Ga$	
12	$\quad\quad \bot$	$\bot$I 11, 10
13	$\quad\quad Fb$	$\bot$E 12
14	$\quad Ga \to Fb$	$\to$E 11–13
15	$\quad \exists x(Ga \to Fx)$	$\exists$I 14
16	$\exists x(Ga \to Fx)$	TND 2–9, 10–15

6. $\exists x(Fx \to Ga), \forall xFx \to Ga$

1	$\exists x(Fx \to Ga)$	
2	$\forall xFx$	
3	$Fb \to Ga$	
4	Fb	$\forall$E 2
5	Ga	$\to$E 3, 4
6	Ga	$\exists$E 1, 3–5
7	$\forall xFx \to Ga$	$\to$I 2–6

1	$\forall xFx \to Ga$	
2	$\forall xFx$	
3	Ga	$\to$E 1, 2
4	Fb	
5	Ga	R 3
6	$Fb \to Ga$	$\to$I 4–5
7	$\exists x(Fx \to Ga)$	$\exists$I 6
8	$\neg\forall xFx$	
9	$\exists x\neg Fx$	CQ 8
10	$\neg Fb$	
11	Fb	
12	$\bot$	$\bot$I 11, 10
13	Ga	$\bot$E 12
14	$Fb \to Ga$	$\to$I 11–13
15	$\exists x(Fx \to Ga)$	$\exists$I 14
16	$\exists x(Fx \to Ga)$	$\exists$E 9, 10–15
17	$\exists x(Fx \to Ga)$	TND 2–7, 8–16

NB: the variable 'x' does not occur in 'Ga'.

When all the quantifiers occur at the beginning of a sentence, that sentence is said to be in PRENEX NORMAL FORM. Together with the CQ rules, these equivalences are sometimes called PRENEXING RULES, since they give us a means for putting any sentence into prenex normal form.

Rules for identity 32

A. Provide a proof of each claim.

1. $Pa \vee Qb, Qb \rightarrow b = c, \neg Pa \vdash Qc$

1	$Pa \vee Qb$	
2	$Qb \rightarrow b = c$	
3	$\neg Pa$	
4	Qb	DS 1, 3
5	$b = c$	→E 2, 4
6	Qc	=E 5, 4

2. $m = n \vee n = o, An \vdash Am \vee Ao$

1	$m = n \vee n = o$	
2	An	
3	$\quad m = n$	
4	$\quad Am$	=E 3, 2
5	$\quad Am \vee Ao$	∨I 4
6	$\quad n = o$	
7	$\quad Ao$	=E 6, 7
8	$\quad Am \vee Ao$	∨I 7
9	$Am \vee Ao$	∨E 1, 3–5, 6–8

3. $\forall x\ x = m, Rma \vdash \exists x Rxx$

1	$\forall x\ x = m$	
2	Rma	
3	$a = m$	∀E 1
4	Raa	=E 3, 2
5	$\exists x Rxx$	∃I 4

4. $\forall x \forall y(Rxy \to x = y) \vdash Rab \to Rba$

1	$\forall x \forall y(Rxy \to x = y)$	
2	$\quad Rab$	
3	$\quad \forall y(Ray \to a = y)$	$\forall$E 1
4	$\quad Rab \to a = b$	$\forall$E 3
5	$\quad a = b$	$\to$E 4, 2
6	$\quad Raa$	$=$E 5, 2
7	$\quad Rba$	$=$E 5, 6
8	$Rab \to Rba$	$\to$I 2–7

5. $\neg \exists x \neg x = m \vdash \forall x \forall y(Px \to Py)$

1	$\neg \exists x \neg x = m$	
2	$\forall x \neg\neg x = m$	CQ 1
3	$\neg\neg a = m$	$\forall$E 2
4	$a = m$	DNE 3
5	$\neg\neg b = m$	$\forall$E 2
6	$b = m$	DNE 5
7	$\quad Pa$	
8	$\quad Pm$	$=$E 3, 7
9	$\quad Pb$	$=$E 5, 8
10	$Pa \to Pb$	$\to$I 7–9
11	$\forall y(Pa \to Py)$	$\forall$I 10
12	$\forall x \forall y(Px \to Py)$	$\forall$I 11

6. $\exists x Jx, \exists x \neg Jx \vdash \exists x \exists y\, \neg x = y$

1	$\exists x Jx$	
2	$\exists x \neg Jx$	
3	$\quad Ja$	
4	$\quad\quad \neg Jb$	
5	$\quad\quad\quad a = b$	
6	$\quad\quad\quad Jb$	$=$E 5, 3
7	$\quad\quad\quad \bot$	$\bot$I 6, 4
8	$\quad\quad \neg a = b$	$\neg$I 5–7
9	$\quad\quad \exists y \neg a = y$	$\exists$I 8
10	$\quad\quad \exists x \exists y \neg x = y$	$\exists$I 9
11	$\quad \exists x \exists y \neg x = y$	$\exists$E 2, 4–10
12	$\exists x \exists y \neg x = y$	$\exists$E 1, 3–11

7. $\forall x(x = n \leftrightarrow Mx), \forall x(Ox \vee \neg Mx) \vdash On$

1	$\forall x(x = n \leftrightarrow Mx)$	
2	$\forall x(Ox \vee \neg Mx)$	
3	$n = n \leftrightarrow Mn$	∀E 1
4	$n = n$	=I
5	Mn	↔E 3, 4
6	$On \vee \neg Mn$	∀E 2
7	$\neg On$	
8	$\neg Mn$	DS 6, 7
9	$\bot$	⊥I 5, 8
10	$\neg\neg On$	¬I 7–9
11	On	DNE 10

8. $\exists x Dx, \forall x(x = p \leftrightarrow Dx) \vdash Dp$

1	$\exists x Dx$	
2	$\forall x(x = p \leftrightarrow Dx)$	
3	Dc	
4	$c = p \leftrightarrow Dc$	∀E 2
5	$c = p$	↔E 4, 3
6	Dp	=E 5, 3
7	Dp	∃E 1, 3–6

9. $\exists x\big[(Kx \wedge \forall y(Ky \rightarrow x = y)) \wedge Bx\big], Kd \vdash Bd$

1	$\exists x\big[(Kx \wedge \forall y(Ky \rightarrow x = y) \wedge Bx\big]$	
2	Kd	
3	$(Ka \wedge \forall y(Ky \rightarrow a = y)) \wedge Ba$	
4	$Ka \wedge \forall y(Ky \rightarrow a = y)$	∧E 3
5	Ka	∧E 4
6	$\forall y(Ky \rightarrow a = y)$	∧E 4
7	$Kd \rightarrow a = d$	∀E 6
8	$a = d$	→E 7, 2
9	Ba	∧E 3
10	Bd	=E 8, 9
11	Bd	∃E 1, 3–10

10. $\vdash Pa \to \forall x(Px \vee \neg x = a)$

1	Pa	
2	$b = a$	
3	Pb	=E 2, 1
4	$Pb \vee \neg b = a$	$\vee$I 3
5	$\neg b = a$	
6	$Pb \vee \neg b = a$	$\vee$I 5
7	$Pb \vee \neg b = a$	TND 2–4, 5–6
8	$\forall x(Px \vee \neg x = a)$	$\forall$I 7
9	$Pa \to \forall x(Px \vee \neg x = a)$	$\to$I 1–8

B. Identity is an EQUIVALENCE RELATION, which means that it is reflexive, symmetric, and transitive:

REFLEXIVITY: $\forall x\, x = x$

1	$a = a$	=I
2	$\forall x\, x = x$	$\forall$I 1

SYMMETRY: $\forall x \forall y(x = y \to y = x)$

1	$a = b$	
2	$b = b$	=I
3	$b = a$	=E 1, 2
4	$a = b \to b = a$	$\to$I 1–3
5	$\forall y(a = y \to y = a)$	$\forall$I 4
6	$\forall x \forall y(x = y \to y = x)$	$\forall$I 5

TRANSITIVITY: $\forall x \forall y \forall z((x = y \wedge y = z) \to x = z)$

1	$a = b \wedge b = c$	
2	$a = b$	$\wedge$E 1
3	$b = c$	$\wedge$E 2
4	$a = c$	=E 2, 3
5	$(a = b \wedge b = c) \to a = c$	$\to$I 1–4
6	$\forall z((a = b \wedge b = z) \to a = z)$	$\forall$I 5
7	$\forall y \forall z((a = y \wedge y = z) \to a = z)$	$\forall$I 6
8	$\forall x \forall y \forall z((x = y \wedge y = z) \to x = z)$	$\forall$I 7

Show that the reflexivity, symmetry and transitivity of identity are all theorems of FOL.

C. Show that the following are provably equivalent:

- $\exists x\big([Fx \land \forall y(Fy \to x = y)] \land x = n\big)$
- $Fn \land \forall y(Fy \to n = y)$

And hence that both have a decent claim to symbolise the English sentence 'Nick is the F'.

In one direction:

1	$\exists x\big([Fx \land \forall y(Fy \to x = y)] \land x = n\big)$	
2	$\quad [Fa \land \forall y(Fy \to a = y)] \land a = n$	
3	$\quad a = n$	$\land$E 2
4	$\quad Fa \land \forall y(Fy \to a = y)$	$\land$E 2
5	$\quad Fa$	$\land$E 4
6	$\quad Fn$	=E 3, 5
7	$\quad \forall y(Fy \to a = y)$	$\land$E 4
8	$\quad \forall y(Fy \to n = y)$	=E 3, 7
9	$\quad Fn \land \forall y(Fy \to n = y)$	$\land$I 6, 8
10	$Fn \land \forall y(Fy \to n = y)$	$\exists$E 1, 2–9

And now in the other:

1	$Fn \land \forall y(Fy \to n = y)$	
2	$n = n$	=I
3	$[Fn \land \forall y(Fy \to n = y)] \land n = n$	$\land$I 1, 2
4	$\exists x\big([Fx \land \forall y(Fy \to x = y)] \land x = n\big)$	$\exists$I 3

D. In §22, we said that the following are logically equivalent symbolisations of the English sentence 'there is exactly one F':

- $\exists x Fx \land \forall x \forall y\big[(Fx \land Fy) \to x = y\big]$
- $\exists x\big[Fx \land \forall y(Fy \to x = y)\big]$
- $\exists x \forall y(Fy \leftrightarrow x = y)$

Show that they are all provably equivalent. (*Hint*: to show that three claims are provably equivalent, it suffices to show that the first proves the second, the second proves the third and the third proves the first; think about why.)

It suffices to show that the first proves the second, the second proves the third and the third proves the first, for we can then show that any of them prove any others, just by chaining the proofs together (numbering lines, where necessary. Armed with this, we start on the first proof:

1	$\exists x Fx \land \forall x \forall y\big[(Fx \land Fy) \to x = y\big]$	
2	$\exists x Fx$	$\land$E 1
3	$\forall x \forall y\big[(Fx \land Fy) \to x = y\big]$	$\land$E 1
4	$\mid\ Fa$	
5	$\mid\ \forall y\big[(Fa \land Fy) \to a = y\big]$	$\forall$E 3
6	$\mid\ (Fa \land Fb) \to a = b$	$\forall$E 5
7	$\mid\ \mid\ Fb$	
8	$\mid\ \mid\ Fa \land Fb$	$\land$I 4, 7
9	$\mid\ \mid\ a = b$	$\to$E 6, 8
10	$\mid\ Fb \to a = b$	$\to$I 7–9
11	$\mid\ \forall y(Fy \to a = y)$	$\forall$I 10
12	$\mid\ Fa \land \forall y(Fy \to a = y))$	$\land$I 4, 11
13	$\mid\ \exists x\big[Fx \land \forall y(Fy \to x = y)\big]$	$\exists$I 12
14	$\exists x\big[Fx \land \forall y(Fy \to x = y)\big]$	$\exists$E 2, 4–13

Now for the second proof:

1	$\exists x\big[Fx \land \forall y(Fy \to x = y)\big]$	
2	$\mid\ Fa \land \forall y(Fy \to a = y)$	
3	$\mid\ Fa$	$\land$E 2
4	$\mid\ \forall y(Fy \to a = y)$	$\land$E 2
5	$\mid\ \mid\ Fb$	
6	$\mid\ \mid\ Fb \to a = b$	$\forall$E 4
7	$\mid\ \mid\ a = b$	$\to$E 6, 5
8	$\mid\ \mid\ a = b$	
9	$\mid\ \mid\ Fb$	$=$E 8, 3
10	$\mid\ Fb \leftrightarrow a = b$	$\leftrightarrow$I 5–7, 8–9
11	$\mid\ \forall y(Fy \leftrightarrow a = y)$	$\forall$I 10
12	$\mid\ \exists x \forall y(Fy \leftrightarrow x = y)$	$\exists$I 11
13	$\exists x \forall y(Fy \leftrightarrow x = y)$	$\exists$E 1, 2–12

And finally, the third proof:

1	$\exists x \forall y(Fy \leftrightarrow x = y)$	
2	$\quad \forall y(Fy \leftrightarrow a = y)$	
3	$\quad Fa \leftrightarrow a = a$	$\forall$E 2
4	$\quad a = a$	=I
5	$\quad Fa$	$\leftrightarrow$E 3, 4
6	$\quad \exists x Fx$	$\exists$I 5
7	$\quad\quad Fb \wedge Fc$	
8	$\quad\quad Fb$	$\wedge$E 7
9	$\quad\quad Fb \leftrightarrow a = b$	$\forall$E 2
10	$\quad\quad a = b$	$\leftrightarrow$E 9, 8
11	$\quad\quad Fc$	$\wedge$E 7
12	$\quad\quad Fc \leftrightarrow a = c$	$\forall$E 2
13	$\quad\quad a = c$	$\leftrightarrow$E 12, 11
14	$\quad\quad b = c$	=E 10, 13
15	$\quad (Fb \wedge Fc) \rightarrow b = c$	$\rightarrow$I 8–14
16	$\quad \forall y\big[(Fb \wedge Fy) \rightarrow b = y\big]$	$\forall$I 15
17	$\quad \forall x \forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$	$\forall$I 16
18	$\quad \exists x Fx \wedge \forall x \forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$	$\wedge$I 6, 17
19	$\exists x Fx \wedge \forall x \forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$	$\exists$E 1, 2–18

E. Symbolise the following argument

> There is exactly one F. There is exactly one G. Nothing is both F and G. So: there are exactly two things that are either F or G.

And offer a proof of it.

Here's the symbolisation, the proof will come over the page:

$\exists x\big[Fx \wedge \forall y(Fy \rightarrow x = y)\big]$,
$\exists x\big[Gx \wedge \forall y(Gy \rightarrow x = y)\big]$,
$\forall x(\neg Fx \vee \neg Gx)$ $\therefore$
$\exists x \exists y\big[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))\big]$

1	$\exists x\big[Fx \wedge \forall y(Fy \rightarrow x = y)\big]$	
2	$\exists x\big[Gx \wedge \forall y(Gy \rightarrow x = y)\big]$	
3	$\forall x(\neg Fx \vee \neg Gx)$	
4	$Fa \wedge \forall y(Fy \rightarrow a = y)$	
5	Fa	$\wedge$E 4
6	$\forall y(Fy \rightarrow a = y)$	$\wedge$E 4
7	$\neg Fa \vee \neg Ga$	$\forall$E 3
8	$\neg Ga$	DS 7, 5
9	$Gb \wedge \forall y(Gy \rightarrow b = y)$	
10	Gb	$\wedge$E 9
11	$\forall y(Gy \rightarrow b = y)$	$\wedge$E 9
12	$a = b$	
13	Ga	=E 12, 10
14	$\perp$	$\perp$I 13, 8
15	$\neg a = b$	$\neg$I 12–14
16	$Fc \vee Gc$	
17	Fc	
18	$Fc \rightarrow a = c$	$\forall$E 6
19	$a = c$	$\rightarrow$E 18, 17
20	$a = c \vee b = c$	$\vee$I 19
21	Gc	
22	$Gc \rightarrow b = c$	$\forall$E 11
23	$b = c$	$\rightarrow$E 22, 21
24	$a = c \vee b = c$	$\vee$I 23
25	$a = c \vee b = c$	$\vee$E 16, 17–20, 21–24
26	$(Fc \vee Gc) \rightarrow (a = c \vee b = c)$	$\rightarrow$I 16–25
27	$\forall z((Fz \vee Gz) \rightarrow (a = z \vee b = z))$	$\forall$I 26
28	$\neg a = b \wedge \forall z((Fz \vee Gz) \rightarrow (a = z \vee b = z))$	$\wedge$I 15, 27
29	$\exists y\big[\neg a = y \wedge \forall z((Fz \vee Gz) \rightarrow (a = z \vee y = z))\big]$	$\exists$I 28
30	$\exists x \exists y\big[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))\big]$	$\exists$I 29
31	$\exists x \exists y\big[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))\big]$	$\exists$E 2, 9–30
32	$\exists x \exists y\big[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))\big]$	$\exists$E 1, 4–31