The Foundations of Mathematics
Lecture Five

Gödel's Incompleteness Theorems

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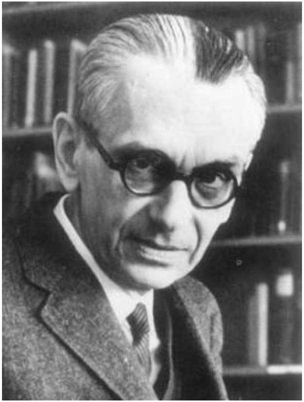
Kurt Gödel

- Gödel is one of the most important intellectual figures of the 20th Century
- His Incompleteness Theorems were monumental achievements in mathematical logic
- They had important technical consequences for subjects like computer theory
- But they also have huge consequences for the philosophy of mathematics



Kurt Gödel

Gödel's Incompleteness Theorems

What is a Theory?

- In formal logic, a **theory** is any deductively closed set of sentences

- What does it mean to call a set of sentences "deductively closed"?
  - Imagine you have a set of sentences, Γ, and now consider all of the sentences you can logically deduce from Γ
  - Now imagine that you add all of those sentences into the set Γ
  - Γ would then be **deductively closed**

- **Formally:** Γ is deductively closed $\leftrightarrow_{df}$ for every sentence $\mathcal{A}$, $\mathcal{A} \in \Gamma \leftrightarrow \Gamma \vdash \mathcal{A}$

An Example

- Imagine we start with the set of the following set of sentences

$$\left\{ \begin{array}{c} \text{`Frege was a logicist'} \\ \text{`Russell was a logicist'} \end{array} \right\}$$

- We can deduce lots of sentences from this set:
  - Frege was a logicist $\leftrightarrow$ Russell was a logicist
  - $\exists x$(x was a logicist)
  - ...
- We can make a theory by **deductively closing** this set
  - The *deductive closure* of $\Gamma =_{df} \{\mathcal{A} : \Gamma \vdash \mathcal{A}\}$
- We call the sentences in a theory the **theorems** of that theory

Property One: ω-Consistency

- One of the most important properties a theory can have is **consistency**
  - Γ is consistent $\leftrightarrow_{df}$ there is no sentence $\mathcal{A}$ such that $\Gamma \vdash \mathcal{A}$ and $\Gamma \vdash \neg\mathcal{A}$
- But when dealing with arithmetical theories, we would also probably demand **ω-consistency**
  - Γ is ω-consistent $\leftrightarrow_{df}$ if $\Gamma \vdash \neg\mathcal{A}(n)$ for each numeral n, then $\Gamma \nvdash \exists x\mathcal{A}(x)$
- Every ω-consistent theory of arithmetic is consistent, but not *vice versa*
- However, any **true** theory of arithmetic should be ω-consistent!

Property Two: Negation-Completeness

- Another good property for a theory is **negation-completeness**
  - Γ is negation-complete $\leftrightarrow_{df}$ for every sentence, $\mathcal{A}$, in the language of Γ, either $\Gamma \vdash \mathcal{A}$ or $\Gamma \vdash \neg\mathcal{A}$
- If a theory is negation complete, then it **decides** every sentence in the language of the theory:
  - it either proves that sentence ($\Gamma \vdash \mathcal{A}$)
  - or it refutes that sentence ($\Gamma \vdash \neg\mathcal{A}$)
- A negation-complete theory leaves nothing out

Axiomatisation

- Theories are **infinite** sets
  - Theories are deductively closed, and a set of sentences always has *infinitely many* deductive implications
- **How could a finite mind possibly comprehend these infinitely complex theories!?**
- **By their axiomatisations!**
  - Θ is an axiom-set for $\Gamma \leftrightarrow_{df} \Gamma =$ the deductive closure of Θ
  - $\mathcal{A}$ is an axiom of Γ relative to axiomatisation $\Theta \leftrightarrow_{df} \Theta$ is an axiom-set for Γ and $\mathcal{A} \in \Theta$
- A finite mind could obviously comprehend *infinite* Γ if it could be axiomatised by some *finite* Θ, but that is not the only way...

Peano Arithmetic

- **Peano Arithmetic** is the deductive closure of the following axioms:

  (1) $\forall x(0 \neq Sx)$
  (2) $\forall x \forall y(Sx = Sy \rightarrow x = y)$

  (3) $\forall x(x + 0 = x)$
  (4) $\forall x \forall y(x + Sy = S(x + y))$

  (5) $\forall x(x \times 0 = 0)$
  (6) $\forall x \forall y(x \times Sy = (x \times y) + x)$

  (7) $\big(\mathcal{A}(0) \wedge \forall x(\mathcal{A}(x) \rightarrow \mathcal{A}(Sx))\big) \rightarrow \forall x \mathcal{A}(x)$

- Axiom (7) is really an axiom **scheme** — *every* instance counts as an axiom

- So Peano Arithmetic has **infinitely many** axioms

Property Three: Recursively Axiomatisable

- Γ **recursively axiomatisable** just in case there is a recursive function f s.t. $f(\mathcal{A}) = 1$ if $\mathcal{A} \in \Gamma$ and $f(\mathcal{A}) = 0$ if $\mathcal{A} \notin \Gamma$
  - That's a slight simplification: we would usually think of f as a function which maps a sentence's *Gödel number* to 0 or 1

- Recursive functions are functions which can be computed by (idealised) computers
  - Last week we defined **primitive recursive** functions
  - All primitive recursive functions are recursive functions, plus functions which search for the least number which meets a certain condition

- **Plausible thought:** A theory can be grasped by a finite mind only if it is recursively axiomatisable

Property Four: Including Robinson Arithmetic

- **Robinson Arithmetic** is the deductive closure of the following axioms:

  (1) $\forall x(0 \neq Sx)$
  (2) $\forall x \forall y(Sx = Sy \rightarrow x = y)$
  (3) $\forall x(x \neq 0 \rightarrow \exists y(x = Sy))$

  (4) $\forall x(x + 0 = x)$
  (5) $\forall x \forall y(x + Sy = S(x + y))$

  (6) $\forall x(x \times 0 = 0)$
  (7) $\forall x \forall y(x \times Sy = (x \times y) + x)$

- **Roughly:** Robinson Arithmetic = Peano Arithmetic − Induction

Gödel's Incompleteness Theorems

All Four Properties Together?

- We've listed four good properties that a theory can have:
  (i) **ω-consistent**
  (ii) **Recursively axiomatisable**
  (iii) **Includes Robinson Arithmetic**
  (iv) **Negation-complete**

- It is natural to wonder whether a theory could have *all four* of these properties

- In particular, we would surely want our theory of arithmetic to have them all!

Gödel's First Incompleteness Theorem

- There is no theory Γ which has all four of the following properties:
  (i) **ω-consistent**
  (ii) **Recursively axiomatisable**
  (iii) **Includes Robinson Arithmetic**
  (iv) **Negation-complete**

Rosser's Strengthening

- There is no theory Γ which has all four of the following properties:
  (i) **Consistent**
  (ii) **Recursively axiomatisable**
  (iii) **Includes Robinson Arithmetic**
  (iv) **Negation-complete**

Gödel Numbering

- Gödel started by introducing a code to let us represent symbols and strings of symbols with numbers
- The number that represents a string of symbols $a_1 \ldots a_n$ is called its **Gödel number**, and is written $\ulcorner a_1 \ldots a_n \urcorner$
  - First assign each primitive symbol a number
  - $\ulcorner a_1 \ldots a_n \urcorner =_{df} \pi_1^{g(a_1)} \times \ldots \times \pi_n^{g(a_n)}$

    (π_i is the ith prime number, and $g(a)$ is the number we assigned to primitive symbol a)
- **The Fundamental Theorem of Arithmetic** — that every number has a unique prime factorisation — guarantees we can decode a string of symbols from its Gödel number

Proof in Γ

- We can also introduce Gödel numbers for **superstrings** — i.e. strings of strings of symbols
  - Let $s_1, \ldots, s_n$ be a superstring of symbols
  - We can code this superstring as $\pi_1^{\ulcorner s_1 \urcorner} \times \ldots \times \pi_n^{\ulcorner s_n \urcorner}$
- A sequence of sentences is a superstring of symbols
- Gödel demonstrated that, if Γ is recursively axiomatisable and includes Robinson Arithmetic, then Γ can define its own proof relation, $Prov_\Gamma(m, n)$
  - $\Gamma \vdash Prov_\Gamma(m, n)$ iff m is the Gödel number of a sequence of sentences, $\mathcal{A}_1, \ldots, \mathcal{A}_n$, and n is the Gödel number of a sentence, $\mathcal{C}$, s.t. $\mathcal{A}_1, \ldots, \mathcal{A}_n$ constitutes a proof of $\mathcal{C}$ in Γ

The Diagonalisation Lemma

- Let Γ be recursively axiomatisable and include Robinson Arithmetic
- In the language of Γ, for each open formula with one free variable, $\mathcal{A}(\chi)$, there is a sentence D s.t.:
  - $\Gamma \vdash \mathcal{A}(\ulcorner D \urcorner) \leftrightarrow D$
- **Informally and roughly:** D says of itself that it satisfies $\mathcal{A}$

A Gödel Sentence for Γ

- Let Γ be recursively axiomatisable and include Robinson Arithmetic
- In the language of Γ, there is a sentence G s.t.:
  - $\Gamma \vdash \neg \exists x Prov_{\Gamma}(x, \ulcorner G \urcorner) \leftrightarrow G$
- **Informally and roughly:** G says of itself that it is not provable in Γ

$\Gamma \nvdash G$

- Gödel proved that if Γ is consistent, then $\Gamma \nvdash G$
- Here is a **rough and informal** argument
  - Suppose $\Gamma \vdash G$
  - In that case, some sequence of sentences is a derivation of G from Γ; let n be the Gödel number of such a sequence
  - It follows that $\Gamma \vdash Prov_{\Gamma}(n, \ulcorner G \urcorner)$
  - Therefore, $\Gamma \vdash \exists x Prov_{\Gamma}(x, \ulcorner G \urcorner)$
  - But we already know that $\Gamma \vdash \neg\exists x Prov_{\Gamma}(x, \ulcorner G \urcorner) \leftrightarrow G$
  - So, since $\Gamma \vdash G$, *modus ponens* yields $\Gamma \vdash \neg\exists x Prov_{\Gamma}(x, \ulcorner G \urcorner)$
  - So Γ is inconsistent

$\Gamma \nvdash \neg G$

- Gödel also proved that if Γ is ω-consistent, then $\Gamma \nvdash \neg G$
- Here is a **rough and informal** argument
  - Suppose $\Gamma \vdash \neg G$
  - We already know that $\Gamma \vdash \neg\exists x Prov_\Gamma(x, \ulcorner G \urcorner) \leftrightarrow G$
  - So by *modus tollens*, $\Gamma \vdash \exists x Prov_\Gamma(x, \ulcorner G \urcorner)$
    - Now suppose that $\Gamma \vdash Prov_\Gamma(n, \ulcorner G \urcorner)$, where n is any numeral you like
    - In that case, $\Gamma \vdash G$
    - It would follow that $\Gamma \vdash G \wedge \neg G$
  - So for each n, $\Gamma \vdash \neg Prov_\Gamma(n, \ulcorner G \urcorner)$
  - So Γ is not ω-consistent

Gödel's First Incompleteness Theorem (Again)

- The upshot is that if Γ has these three properties:
  (i) **ω-consistent**
  (ii) **Recursively axiomatisable**
  (iii) **Includes Robinson Arithmetic**

  then Γ cannot be
  (iv) **Negation-complete**

- If Γ has (i)–(iii), then Gödel can construct his G, such that:
  - $\Gamma \nvdash G$
  - $\Gamma \nvdash \neg G$

Extending Γ?

- What happens if we extend our theory Γ by adding G to it?
- Clearly, $\Gamma \cup \{G\} \vdash G$
- But now we'll be able to make a **new** sentence, G', s.t.:
  - $\Gamma \cup \{G\} \vdash \neg \exists x Prov_{\Gamma \cup \{G\}}(x, \ulcorner G' \urcorner) \leftrightarrow G'$
  - **Informally and roughly:** G' says of itself that it is not provable in $\Gamma \cup \{G\}$
- And then we'll prove:
  - $\Gamma \cup \{G\} \nvdash G'$
  - $\Gamma \cup \{G\} \nvdash \neg G'$
- So $\Gamma \cup \{G\}$ won't be **negation-complete** either!

Extending Γ?

- What happens if we extend our theory Γ by adding $\neg G$ to it?
- Clearly, $\Gamma \cup \{\neg G\} \vdash \neg G$
- But now we'll be able to make a **new** sentence, G'', s.t.:
  - $\Gamma \cup \{\neg G\} \vdash \neg \exists x Prov_{\Gamma \cup \{\neg G\}}(x, \ulcorner G'' \urcorner) \leftrightarrow G''$
  - **Informally and roughly:** G'' says of itself that it is not provable in $\Gamma \cup \{\neg G\}$
- And then we'll prove:
  - $\Gamma \cup \{\neg G\} \nvdash G''$
  - $\Gamma \cup \{\neg G\} \nvdash \neg G''$
- So $\Gamma \cup \{\neg G\}$ won't be **negation-complete** either!

Gödel's Incompleteness Theorems

Gödel's Second Incompleteness Theorem

- Abbreviate $\neg\exists x Prov_{\Gamma}(x, \ulcorner 0 = 1 \urcorner)$ as Con_{Γ}
- If Γ has these three properties:
  (i) **Consistent**
  (ii) **Recursively axiomatisable**
  (iii) **Includes Robinson Arithmetic**

  then Γ cannot prove that Γ is consistent:

  $\Gamma \nvdash \mathrm{Con}_{\Gamma}$

How Did Gödel Do It!?

- If Γ is consistent, recursively axiomatisable, and includes Robinson Arithmetic, then Γ **itself** contains all of the resources needed to prove Gödel's First Incompleteness Theorem:
  - $\Gamma \vdash Con_{\Gamma} \rightarrow \neg\exists x Prov_{\Gamma}(x, \ulcorner G \urcorner)$
- And since G is defined so that $\Gamma \vdash \neg\exists x Prov_{\Gamma}(x, \ulcorner G \urcorner) \leftrightarrow G$, it follows that:
  - $\Gamma \vdash Con_{\Gamma} \rightarrow G$
- Now suppose that $\Gamma \vdash Con_{\Gamma}$; it would follow by *modus ponens* that $\Gamma \vdash G$, violating the First Incompleteness Theorem
- So $\Gamma \nvdash Con_{\Gamma}$

Gödel's Second Incompleteness Theorem (Again)

- Here is a concise statement on the Second Incompleteness Theorem:
  - If Γ is **consistent**, **recursively axiomatisable**, and **includes Robinson Arithmetic**, then it cannot **prove** its own **consistency**
- Or to put it in a slightly more paradoxical way:
  - If a recursively axiomatisable theory which includes Robinson Arithmetic proves its own consistency, then that theory is inconsistent!
- Compare that to this good advice for life:
  - If someone tells you that they are trustworthy, then they are not trustworthy!

Gödel's Incompleteness Theorems

The Philosophical Significance of Gödel's Theorems

- It is hard to over-estimate the philosophical significance of Gödel's theorems
- They have huge consequences for pretty much **every** philosophy of maths
- We will quickly look at the absolutely devastating consequences that they had for Hilbert's Programme

Hilbert's Programme

- Hilbert divided mathematics into two parts: **Finitary Mathematics** (FM) and **Ideal Mathematics** (IM)
- FM is meaningful, but IM is just a game
- Hilbert's Programme was to give a finitary proof that FM + IM is consistent

David Hilbert

The Failure of Hilbert's Programme

- IM must be recursively axiomatisable, and it must include Robinson Arithmetic
- So, if we assume that IM is consistent, the Second Incompleteness Theorem implies:
  - $\mathrm{IM} \nvdash Con_{\mathrm{IM}}$
- FM is a finitary fragment of IM
- So, if FM proved that FM + IM is consistent, IM would prove that IM is consistent
- Therefore, by *modus tollens* , FM does not prove that FM + IM is consistent

Can Hilbert's Programme be Saved?

- **Abandon proof?**
  - Maybe we don't need to *prove* that IM + FM is consistent?
  - Maybe it's just enough if IM + FM is *in fact* consistent?

- **Re-conceive consistency?**
  - Maybe Con_{IM} isn't the best way to formalize the claim that IM is consistent?

- **Abandon recursive axiomatisability?**
  - Maybe FM isn't recursively axiomatisable?
  - Then FM could prove the consistency of IM + FM without violating the Second Incompleteness Theorem

Gödel's Incompleteness Theorems

Syntax vs. Semantics

- **Syntax**
  - When we think about a language *syntactically*, we are thinking of that language as a mere system of signs
  - We do not care what the signs *mean*, or whether they mean anything at all
- **Semantics**
  - When we think about a language *semantically*, we are thinking about the language as a system of symbols with meanings
  - For logical purposes, we are interested in things like: truth-values, references, satisfaction conditions

Two Notions of Entailment

- **Syntactic Deduction**
  - We can write out a proof using sentences in Γ as premises, and ending with $\mathcal{A}$
  - $\Gamma \vdash \mathcal{A}$
  - **Strictly speaking** we should specify which deductive system we are using: $\Gamma \vdash_{\Delta} \mathcal{A}$

- **Semantic Consequence**
  - Any interpretation which makes all of the sentences in Γ true makes $\mathcal{A}$ true too
  - $\Gamma \vDash \mathcal{A}$
  - **Strictly speaking** we should specify which semantics we are using: $\Gamma \vDash_{\Sigma} \mathcal{A}$

Soundness and Completeness

- Deductive system Δ is **sound** relative to semantics Σ iff:

  if $\Gamma \vdash_{\Delta} \mathcal{A}$ then $\Gamma \vDash_{\Sigma} \mathcal{A}$

- Deductive system Δ is **complete** relative to semantics Σ iff:

  if $\Gamma \vDash_{\Sigma} \mathcal{A}$ then $\Gamma \vdash_{\Delta} \mathcal{A}$

- In the dream scenario, our deductive system is sound *and* complete relative to our semantics
- Classical First-Order Logic is sound and complete relative to the standard semantics

Gödel's Theorems are Primarily Syntactic

- Gödel's Incompleteness Theorems are primarily **syntactic**
  (1) If Γ is ω-consistent, recursively axiomatisable and includes Robinson Arithmetic, then there is a sentence G such that: $\Gamma \nvdash G$ and $\Gamma \nvdash \neg G$
  (2) If Γ is consistent, recursively axiomatisable and includes Robinson Arithmetic, then $\Gamma \nvdash Con_\Gamma$

- **IMPORTANT:** These theorems aren't limited to just one deductive system

- They apply to theory Γ, if its background deductive system includes at least classical FOL, and so long as Γ is consistent, recursively axiomatisable and includes Robinson Arithmetic

Gödel's Theorems can become Semantic

- However, if we assume that we are working with a complete deductive system, we can convert these syntactic results into semantic ones:

(1) If Γ is consistent, effectively axiomatisable and includes Robinson Arithmetic, then there is a sentence G such that: $\Gamma \nvDash G$ and $\Gamma \nvDash \neg G$

(2) If Γ is consistent, effectively axiomatisable and includes Robinson Arithmetic, then $\Gamma \nvDash Con_\Gamma$

Non-Standard Arithmetic

- **First-Order Peano Arithmetic** (PA_1)

  (1) $\forall x(0 \neq Sx)$
  (2) $\forall x \forall y(Sx = Sy \rightarrow x = y)$
  (3) $\forall x(x + 0 = x)$
  (4) $\forall x \forall y(x + Sy = S(x + y))$
  (5) $\forall x(x \times 0 = 0)$
  (6) $\forall x \forall y(x \times Sy = (x \times y) + x)$
  (7) $\big(\mathcal{A}(0) \wedge \forall x(\mathcal{A}(x) \rightarrow \mathcal{A}(Sx))\big) \rightarrow \forall x \mathcal{A}(x)$

- By the Second Incompleteness Theorem: $PA_1 \nvdash Con_{PA_1}$

- Since FOL is complete: $PA_1 \nvDash Con_{PA_1}$
  - Some non-standard interpretation makes all of the sentences in $PA_1 \cup \{\neg Con_{PA_1}\}$ true!

Second-Order Logic

- **IMPORTANT:** Some deductive systems are incomplete relative to their semantics
- A few times we have come up against the difference between **First-Order Logic** and **Second-Order Logic**
- FOL lets us use quantifiers like this: $\exists x$ x is a philosopher
  - The variable x is in the position of a **name**, like 'Socrates' or 'Plato'
- SOL also lets us use quantifiers like this: $\exists X\ X(\text{Socrates})$
  - The variable X is in the position of a **predicate**, like 'is a philosopher' or 'is wise'

Second-Order Logic

- According to the **standard semantics** for SOL, every subset of the first-order domain determines a value of the second-order domain

- No recursively axiomatisable deductive system can be sound and complete relative to this standard semantics; the standard deductive system is sound but not complete
  - For every sentence $\mathcal{A}$ and set of sentences Γ: if $\Gamma \vdash \mathcal{A}$ then $\Gamma \vDash \mathcal{A}$
  - There is some sentence $\mathcal{A}$ and some set of sentences Γ such that: $\Gamma \vDash \mathcal{A}$ and $\Gamma \nvdash \mathcal{A}$

- As a result, when we are dealing with a second-order system, we cannot convert Gödel's syntactic results into semantic ones

Categorical Arithmetic

- **Second-Order Peano Arithmetic** (PA_2)
  (1) $\forall x(0 \neq Sx)$
  (2) $\forall x \forall y(Sx = Sy \rightarrow x = y)$
  (3) $\forall x(x + 0 = x)$
  (4) $\forall x \forall y(x + Sy = S(x + y))$
  (5) $\forall x(x \times 0 = 0)$
  (6) $\forall x \forall y(x \times Sy = (x \times y) + x)$
  (7) $\forall Y\big(\big(Y(0) \wedge \forall x(Y(x) \rightarrow Y(Sx))\big) \rightarrow \forall x Y(x)\big)$
- On the standard semantics PA_2 is **categorical**, meaning all of its models are isomorphic
- It follows that PA_2 is **semantically complete**
  - Either $PA_2 \vDash \mathcal{A}$, or $PA_2 \vDash \neg\mathcal{A}$
- $PA_2 \nvdash Con_{PA_2}$ but $PA_2 \vDash Con_{PA_2}$

For the Seminar

- In the seminar we are going to look at Gödel's own platonist philosophy of mathematics

- Required reading:
  - Gödel, 'What is Cantor's Continuum Problem?', in *B&P*

- It may also be helpful to look at the following secondary material on Gödel's Theorems:
  - Giaquinto, *The Search for Certainty*, Part V