Intermediate Logic Spring
Lecture Eight

Intuitionism and Semantics

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Intuitionism and Semantics

Restricting Classical Logic

- IL is a **restriction** of classical FOL
- The natural deduction system for IL includes all of the basic rules for FOL, **apart from TND**

i	$\quad\vert\ \mathcal{A}$	
j	$\quad\vert\ \mathcal{B}$	
k	$\quad\vert\ \neg\mathcal{A}$	
l	$\quad\vert\ \mathcal{B}$	
	$\mathcal{B}$	TND, i–j, k–l

Rejecting Derived Rules of FOL

- If we reject the basic rule of TND, then we have to reject a number of derived rules too
- Most obviously, we have to reject DNE:

m	$\neg\neg\mathcal{A}$	
	$\mathcal{A}$	DNE, m

Rejecting Derived Rules of FOL

- We also have to reject one of the De Morgan Rules

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

- But we get to keep the other three De Morgan Rules!

Rejecting Derived Rules of FOL

- We also have to reject one of the rules for Converting Quantifiers

m	$\neg\forall x\mathcal{A}$	
	$\exists x\neg\mathcal{A}$	CQ, m

- But we get to keep the other three rules for Converting Quantifiers!

Arguing for Intuitionism

- Last week we looked at a **proof-theoretic** argument for IL
- This week we will look at the **semantics** for IL
- We will then look at an argument for IL developed by Michael Dummett
- According to this argument, we should prefer the semantics for IL over the classical semantics (at least for mathematical discourse), and so should accept IL

Intuitionism and Semantics

What is the Fundamental Semantic Concept?

- Every semantic theory we have looked at so far has taken TRUTH to be the fundamental semantic concept
- **Guiding Idea:** To understand a sentence is to know its truth-conditions
- The semantic theories gave us a way of displaying the truth-conditions of the sentences in the language at hand
  - $\mathcal{A} \wedge \mathcal{B}$ is **true** iff $\mathcal{A}$ is true and $\mathcal{B}$ is true
  - $\Box\mathcal{A}$ is **true** iff $\mathcal{A}$ is true at every world
  - $\forall \chi \mathcal{A}(\chi)$ is **true** iff $\mathcal{A}(c)$ is true, no matter what object in the domain is named by c
  - $\forall \mathcal{X} \mathcal{A}(\mathcal{X})$ is **true** iff $\mathcal{A}(\mathcal{F})$ is true, no matter what subset of the domain is the extension of $\mathcal{F}$

What is the Fundamental Semantic Concept?

- Many intuitionists reject the assumption that TRUTH is the fundamental semantic concept
- They take WARRANTED ASSERTIBILITY to the be the fundamental semantic concept instead
- **Guiding Idea:** To understand a sentence is to know under what circumstances you would be warranted to assert it
- Once you have told me when I would be warranted to assert a sentence, you have told me everything there is to know about what that sentence means

Two Different Perspectives on Language

- **The fundamental semantic concept is TRUTH**
  - When we take this approach, we are thinking of language as fundamentally *representational*
  - The distinctive thing about language is that sentences *represent* the world, truly or falsely

- **The fundamental semantic concept is WARRANTED ASSERTIBILITY**
  - When we take this approach, we are thinking of language as fundamentally something we *use* for various purposes
  - Our semantic theory needs to tell us the rules for how to use a given sentence

Intuitionism and Semantics

When are You Warranted to Assert a Sentence?

- **Question:** What does it mean to say that we are warranted to assert a given sentence?
- There is probably no single answer to this question; different areas of discourse seem to be governed by different rules for assertion
- In scientific contexts, you cannot assert a sentence unless you have good (experimental or theoretical) evidence in its favour
- In everyday contexts, you are sometimes allowed to assert things for which you have no more backing than your own memory

From Assertibility to Provability

- There is one area of discourse where the rules of assertion seem pretty clear
  - In **mathematics**, you cannot assert a sentence unless you have a *proof* for it

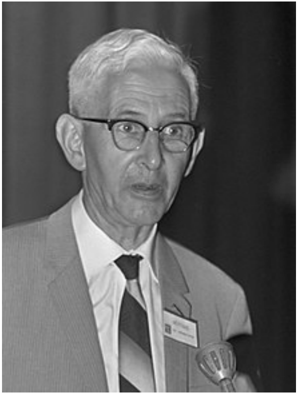
- So for intuitionists, if we are presenting a semantics for a mathematical language, we should take PROOF to be our fundamental notion

- Once you have told me what it would take to *prove* a mathematical sentence, you have told me everything there is to know about what that sentence means

  (**Remember:** Intuitionism started life as a philosophy of mathematics)

Introducing the BHK Semantics

- This kind of semantics was rigorously developed by Heyting, and independently by Kolmogorov
- It is now known as the BHK semantics
  - The extra ‘B’ is for Brouwer, the inventor of intuitionism
- The semantics gives us a method of describing proofs of more complex sentences in terms of proofs of simpler sentences



Arend Heyting

Intuitionism and Semantics

(1) A proof of $\mathcal{A} \wedge \mathcal{B}$ consists of a proof of $\mathcal{A}$ is and a proof of $\mathcal{B}$

(2) A proof of $\mathcal{A} \vee \mathcal{B}$ consists of a proof of $\mathcal{A}$ or a proof of $\mathcal{B}$

(3) A proof of $\mathcal{A} \to \mathcal{B}$ consists of a method for converting any proof of $\mathcal{A}$ into a proof of $\mathcal{B}$

(4) A proof of $\mathcal{A} \leftrightarrow \mathcal{B}$ consists of a proof of $\mathcal{A} \to \mathcal{B}$ and a proof of $\mathcal{B} \to \mathcal{A}$

(5) A proof of $\neg\mathcal{A}$ consists of a proof of $\mathcal{A} \to \bot$

(6) A proof of $\exists \chi \mathcal{A}(\chi)$ consists of a proof of $\mathcal{A}(c)$, for some element of the domain, c

(7) A proof of $\forall \chi \mathcal{A}(\chi)$ consists of a method which acts on any element in the domain, c, and delivers a proof that $\mathcal{A}(c)$

Intuitionism and Semantics

BHK and LEM

- Let's take G to be a statement of *Goldbach's Conjecture*
  - **Goldbach's Conjecture:** Every even number greater than 2 is the sum of two primes
- According to the BHK semantics, a proof of $G \vee \neg G$ would consist in a proof of G or a proof of $\neg G$
- As things stand, no one has a proof of either of these
- So, assuming that a sentence is assertible iff it is provable, no one is in a position to assert $G \vee \neg G$

BHK and LEM

- Even though no one actually knows how to prove G or $\neg G$, maybe there is an ideal proof ‘out there’ for one of these sentences
- If so, $G \vee \neg G$ would still be provable in an idealised sense, and so in that sense still assertible
- But what *logical guarantee* do we have that there is a proof of G or of $\neg G$ waiting to be discovered?
- Absent such a guarantee, we have no right to assert $G \vee \neg G$
  - × G is neither provable nor refutable — i.e. $\neg(G \vee \neg G)$
  - ✓ We should remain silent on G

An Instance of Goldbach's Conjecture

- Consider the following *instance* of Goldbach's Conjecture:
  - If 2^{100} is an even number greater than 2, then it is the sum of two primes
- 2^{100} is an absolutely huge even number, and I have no idea if anyone has ever checked whether it is the sum of two primes
- But I know that we **could** check if we liked
  - Just go through all of the primes smaller than 2^{100}, and see if any pair of them add up to 2^{100}
- **Technical Terminology:** It is *decidable* whether 2^{100} is the sum of two primes
  - We have a finite procedure for proving or refuting the claim that 2^{100} is the sum of two primes

An Instance of Goldbach's Conjecture

- Since it is decidable whether 2^{100} is the sum of two primes, the BHK semantics tells us that this instance of LEM is provable:

$$\begin{array}{c} 2^{100} \text{ is the sum of two primes} \\ \vee \\ \neg(2^{100} \text{ is the sum of two primes}) \end{array}$$

- We do not currently have a proof of either disjunct, but we know that one of the disjuncts is **provable**
- As a result, the disjunction is also provable, and is thus **assertible**

An Infinite Generalisation

- The same goes for **every** instance of Goldbach's Conjecture: LEM holds for every single instance
- We only lose our guarantee for LEM when we stop considering instances, and look at the fully general version of Goldbach's Conjecture
  - **Goldbach's Conjecture:** *Every* even number greater than 2 is the sum of two primes
- There are infinitely many even numbers, and as a result, we cannot go through each even number and check whether it is the sum of two primes

An Infinite Generalisation

- Of course, if we start checking even numbers, then we might find one which is not the sum of two primes
- That would be great, because it would amount to a proof of $\neg G$
- But it may be that every even number we check *is* the sum of two primes
- In that case, we would keep proving instance after instance of G, but that would not add up to a proof of G itself
  - No matter how far through the even numbers we go, there might always be a counterexample to Goldbach's Conjecture waiting around the corner

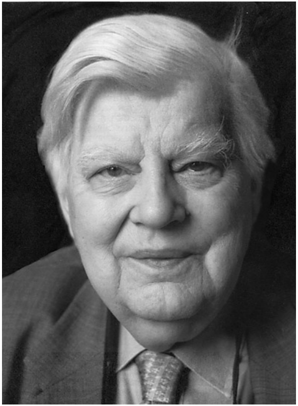
Finite versus Infinite Domains

- Let $\mathcal{F}$ be a **decidable predicate**
  - We have a finite procedure that we can apply to each object in the domain, and each time it either proves that the object satisfies $\mathcal{F}$, or it proves that the object does not satisfy $\mathcal{F}$
- When we are dealing with *finite* domains, this is enough to guarantee that it is decidable whether $\forall x \mathcal{F} x$
  - We will either be able to prove $\forall x \mathcal{F} x$, or we will be able to prove $\neg \forall x \mathcal{F} x$
  - Either way, we will be able to prove $\forall x \mathcal{F} x \vee \neg \forall x \mathcal{F} x$
- When we are dealing with *infinite domains*, this is not enough to guarantee that it is decidable whether $\forall x \mathcal{F} x$
  - We may not have a finite procedure for proving either $\forall x \mathcal{F} x$ or $\neg \forall x \mathcal{F} x$
  - As a result, we may not have a finite procedure for proving $\forall x \mathcal{F} x \vee \neg \forall x \mathcal{F} x$

Intuitionism and Semantics

Dummett's Semantic Arguments

- Dummett is one of the most important advocates of IL there has ever been
- Dummett argued for IL by arguing that the fundamental semantic notion is WARRANTED ASSERTIBILITY, not TRUTH
- Dummett presented two related arguments
  - The Manifestation Argument
  - The Acquisition Argument
- We will focus on the Manifestation Argument



Michael Dummett

The Form of the Manifestation Argument

- The heart of Dummett's argument is a *reductio ad absurdum*
  - Dummett starts off by assuming that TRUTH is the fundamental semantic concept
  - He then attempts to derive an absurd result from this assumption
  - He ends the *reductio* by concluding that TRUTH is **not** the fundamental semantic concept
- After the *reductio*, Dummett simply proposes that ASSERTIBILITY would be a more fruitful candidate for the fundamental semantic concept

From Understanding to Knowing Truth-Conditions

- Assume that TRUTH is the fundamental semantic concept
- Presumably that means that to understand a sentence, you must know its truth-conditions
- For example, to understand Goldbach's Conjecture, G, is to know the conditions under which it would be true

Manifesting your Understanding

- **Dummett's Big Idea:** Whatever exactly your understanding of sentence *s* consists in, it is essential that this understanding be fully *manifestable*
  - You must be able to demonstrate that you have the knowledge which would underlie an understanding of *s*
- This is crucial because meaning is fundamentally **public**
  - The meaning of our sentences is what we communicate to each other, and so it must be possible to make that meaning publicly available
- So if our understanding of *s* consists in knowing its truth-conditions, then we must have some way of manifesting that knowledge

Verifiable Truth-Conditions

- Your understanding of 'Rob is talking' consists in your knowing that 'Rob is talking' is true (now) iff Rob is talking (now)
- You can manifest this knowledge by asserting 'Rob is talking' when I am talking, and asserting its negation when I am not
- The crucial point: There is no problem with taking TRUTH to be the fundamental semantic concept when we are dealing with sentences whose truth-conditions are **verifiable**

Verification-Transcendent Truth-Conditions

- Let G be our statement of **Goldbach Conjecture**: every even number greater than 2 is the sum of two primes
- Suppose that G is undecidable: no finite procedure will ever prove G or $\neg G$
- In that case, the truth-conditions for G are **verification-transcendent**
  - It is beyond our means to verify whether G is true
- **Dummett's Question:** How would you ever *manifest* your knowledge of the *verification-transcendent* truth-conditions for G?

Stating Truth-Conditions

- You might think that you could manifest your knowledge of G's truth-conditions by *explicitly stating what they are*:
  - G is true iff: for all $n > 2$, if n is a multiple of 2 then there are some numbers, j and k, such that each of these numbers is only divisible by 1 and itself, and $n = j + k$

- The trouble with this strategy is that you end up **using** a sentence which has verification-transcendent truth-conditions to state G's verification-transcendent truth-conditions

- This will not be much help to you if you were worried about how you could manifest an understanding of sentences with verification-transcendent truth-conditions!

Manifestation through Use

- Ultimately, your understanding of a sentence can only be manifested through the way that you actually **use** it

- So if to understand G is to know its truth-conditions, then you must be able to manifest this knowledge through the way that you use G

- But what use could manifest knowledge of **verification-transcendent** truth-conditions?

Replacing Truth with Proof

- Dummett does not think that there is any good answer to this question, and so concludes that TRUTH cannot be the fundamental semantic concept

  (Or at least, it can't be in mathematical discourse)

- Dummett recommends that we take PROOF as our fundamental semantic concept for mathematical discourses

- While the *truth-conditions* for G may be verification-transcendent, the *proof-conditions* are not
  - We know what it would take to prove G, and we can manifest that knowledge in various ways
  - For example, we can look over putative proofs of G, and judge whether they are successful

Tomorrow's Seminar

- For tomorrow's seminar, please read:
  - *An Intuitionistic Logic Primer*, §§5–6
  - Michael Dummett, 'The Philosophical Basis of Intuitionsitic Logic'
- The paper by Dummett is one of the places where he develops his Manifestation and Acquisition Arguments for IL

Next Week's Lecture and Seminar

- Next week, we will look at Fitch's Paradox, which uses the resources of Modal Logic and Second-Order Logic to argue against Intuitionistic Logic!
- **Required Reading**
  - Timothy Williamson, 'Intuitionism Disproved?'
  - Dorothy Edgington, 'The Paradox of Knowability'
  - Michael Dummett, 'Victor's Error'
- All three of these articles are very short, and they are all available via the Reading List on the VLE