Intermediate Logic Spring
Week Five

Second-Order Logic

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Introducing Second-Order Logic — The Quantifiers

Two Arguments

Bertrand is a logician

Bertrand is a mathematician

∴ Someone is both a logician and a mathematician

Bertrand is a philosopher

Alfred is a philosopher

∴ Bertrand and Alfred have something in common

Two Arguments

Lb

Mb

$\therefore \exists x(Lx \wedge Mx)$

Bertrand is a philosopher

Alfred is a philosopher

$\therefore$ Bertrand and Alfred have something in common

Two Arguments

$$
\begin{array}{rl}
 & Lb \\
 & Mb \\
\therefore & \exists x(Lx \wedge Mx)
\end{array}
$$

$$
\begin{array}{rl}
 & Pb \\
 & Pa \\
\therefore & \exists X(Xb \wedge Xa)
\end{array}
$$

Introducing Second-Order Logic

- When we introduce variables that go where predicates go, we take the step from first-order logic (FOL) to *second-order logic* (SOL)

- The quantifiers in FOL let us quantify over **objects**
  - '$\exists x(Lx \wedge Mx)$' says that there is some *object* which is both L and M

- The quantifiers in SOL let us quantify over **properties**
  - '$\exists X(Xb \wedge Xa)$' says that there is some *property* which b and a both have

Second-Order Logic

An Extension of FOL

- To get from FOL to SOL, all we need to do add some "second-order variables"
  - A **first-order variable** is a variable which can go where *names* go
  - A **second-order variable** is a variable which can go where *predicates* go

- We will use capital letters from S to Z, with or without numerical subscripts, as our second-order variables
  - S, T, U, V, W, X, Y, Z
  - $S_1, T_1, U_1, V_1, W_1, X_1, Y_1, Z_1$
  - $S_2, T_2, U_2, V_2, W_2, X_2, Y_2, Z_2$
  - ...

Monadic SOL

- As you already know, predicates can have different **adicities**
  - A **monadic** predicate combines with one term at a time
  - A **dyadic** predicate combines with two terms at a time
  - An ***n*-adic** predicate combines with *n* terms at a time

- We *can* divide second-order variables up in exactly the same way...

- ...but to keep things simple, we will require that *all* the second-order variables are **monadic**
  - Second-order variables can only combine with one term at a time

Formulas and Sentences of SOL

- You can build a new formula out of an old one by replacing monadic predicates with second-order variables

$$\neg Fa \Rightarrow \neg Xa$$

$$\forall x \forall y(Px \leftrightarrow Py) \Rightarrow \forall x \forall y(Yx \leftrightarrow Py)$$

$$\forall x \forall y(Px \leftrightarrow Py) \Rightarrow \forall x \forall y(Yx \leftrightarrow Yy)$$

- You can build a new formula out of an old one by binding free second-order variables with quantifiers

$$\neg Xa \Rightarrow \forall X \neg Xa$$

$$\forall x \forall y(Yx \leftrightarrow Yy) \Rightarrow \exists Y \forall x \forall y(Yx \leftrightarrow Yy)$$

- A **sentence** of SOL is just a formula which contains no free variables (first-order or second-order)

Second-Order Logic

An Extension of FOL

- The natural deduction system for SOL is an extension of the system for FOL
  - SOL includes *all* of the rules of FOL, basic and derived
- SOL simply adds some extra rules to govern the *second-order quantifiers*
  - **Second-order quantifiers** are quantifiers which bind second-order variables
- We start with their introduction and elimination rules, which look pretty much exactly the same as the rules for the **first-order quantifiers**

First-Order Existential Introduction

m	$\mathcal{A}(\ldots c \ldots c \ldots)$	
	$\exists \chi \mathcal{A}(\ldots \chi \ldots c \ldots)$	$\exists_1$I, m

- $\mathcal{A}(\ldots c \ldots c \ldots)$ is a sentence containing **one or more** occurrences of the name c
- χ can be any first-order variable that does **not** occur in $\mathcal{A}(\ldots c \ldots c \ldots)$
- $\mathcal{A}(\ldots \chi \ldots c \ldots)$ is the result of replacing **one or more** of the occurrences of c in $\mathcal{A}(\ldots c \ldots c \ldots)$ with χ

Second-Order Existential Introduction

m	$\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$	
	$\exists\mathcal{X}\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{F}\ldots)$	$\exists_2$I, m

- $\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$ is a sentence containing **one or more** occurrences of a monadic predicate $\mathcal{F}$
- $\mathcal{X}$ can be any second-order variable that does **not** occur in $\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$
- $\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{F}\ldots)$ is the result of replacing **one or more** of the occurrences of $\mathcal{F}$ in $\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$ with $\mathcal{X}$

$Pb, Pa \vdash_2 \exists X(Xb \wedge Xa)$

1	Pb	
2	Pa	
3	$Pb \wedge Pa$	$\wedge$I, 1, 2
4	$\exists X(Xb \wedge Xa)$	$\exists_2$I, 3

First-Order Existential Elimination

m	$\exists\chi\mathcal{A}(\ldots\chi\ldots\chi\ldots)$		
n		$\mathcal{A}(\ldots c\ldots c\ldots)$	
o		$\mathcal{B}$	
	$\mathcal{B}$		$\exists_1$E, m, n–o

- c **must not** occur in any undischarged assumptions above line m (including the premises of the argument)
- c **must not** occur in $\exists\chi\mathcal{A}(\ldots\chi\ldots\chi\ldots)$
- c **must not** appear in $\mathcal{B}$

Second-Order Existential Elimination

m	$\exists \mathcal{X}\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$	
n	$\quad \mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$	
o	$\quad \mathcal{B}$	
	$\mathcal{B}$	$\exists_2$E, m, n–o

- $\mathcal{F}$ **must not** occur in any undischarged assumptions above line m (including the premises of the argument)
- $\mathcal{F}$ **must not** occur in $\exists \mathcal{X}\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$
- $\mathcal{F}$ **must not** appear in $\mathcal{B}$

$\exists X(Xa \wedge Xb), \forall Y(Yb \rightarrow \neg Yc) \vdash_2 \exists Z(Za \wedge \neg Zc)$

1	$\exists X(Xa \wedge Xb)$	
2	$\forall Y(Yb \rightarrow \neg Yc)$	
3	$\quad Fa \wedge Fb$	
4	$\quad Fa$	$\wedge$E, 3
5	$\quad Fb$	$\wedge$E, 3
6	$\quad Fb \rightarrow \neg Fc$	$\forall_2$E, 2
7	$\quad \neg Fc$	$\rightarrow$E, 6, 5
8	$\quad Fa \wedge \neg Fc$	$\wedge$I, 4, 7
9	$\quad \exists Z(Za \wedge \neg Zc)$	$\exists_2$I, 8
10	$\exists Z(Za \wedge \neg Zc)$	$\exists_2$E, 1, 3–9

First-Order Universal Introduction

m	$\mathcal{A}(\ldots c \ldots c \ldots)$	
	$\forall \chi \mathcal{A}(\ldots \chi \ldots \chi \ldots)$	$\forall_1$I, m

- $\mathcal{A}(\ldots c \ldots c \ldots)$ is a sentence containing one or more occurrences of the name c, and $\mathcal{A}(\ldots \chi \ldots \chi \ldots)$ is the formula that you get when you replace **all** of those occurrences of c with the first-order variable χ
- c **must not** occur in any undischarged assumptions above line m (including the premises of the argument)
- c **must not** occur in $\forall \chi \mathcal{A}(\ldots \chi \ldots \chi \ldots)$

Second-Order Universal Introduction

m	$\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$	
	$\forall\mathcal{X}\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$	$\forall_2$I, m

- $\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$ is a sentence containing one or more occurrences of the monadic predicate $\mathcal{F}$, and $\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$ is the formula that you get when you replace **all** of those occurrences of $\mathcal{F}$ with the second-order variable $\mathcal{X}$
- $\mathcal{F}$ **must not** occur in any undischarged assumptions above line m (including the premises of the argument)
- $\mathcal{F}$ **must not** occur in $\forall\mathcal{X}\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$

$a = b \vdash_2 \forall W \forall y(y = b \to (Wa \to Wy))$

1	$a = b$	
2	$\quad c = b$	
3	$\quad\quad Fa$	
4	$\quad\quad Fb$	$=$E, 1, 3
5	$\quad\quad Fc$	$=$E, 2, 4
6	$\quad Fa \to Fc$	$\to$I, 3–5
7	$c = b \to (Fa \to Fc)$	$\to$I, 2–6
8	$\forall y(y = b \to (Fa \to Fy))$	$\forall_1$I, 7
9	$\forall W \forall y(y = b \to (Wa \to Wy))$	$\forall_2$I, 8

First-Order Universal Elimination

m	$\forall\chi\mathcal{A}(\ldots\chi\ldots\chi\ldots)$	
	$\mathcal{A}(\ldots c\ldots c\ldots)$	$\forall_1$E, m

- $\mathcal{A}(\ldots\chi\ldots\chi\ldots)$ is a formula containing **one or more** occurrences of some first-order variable χ
- c can be **any** name you like
- $\mathcal{A}(\ldots c\ldots c\ldots)$ is the result of replacing **all** of the occurrences of χ in $\mathcal{A}(\ldots\chi\ldots\chi\ldots)$ with c

Second-Order Universal Elimination

m	$\forall \mathcal{X}\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$	
	$\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$	$\forall_2$E, m

- $\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$ is a formula containing **one or more** occurrences of some second-order variable $\mathcal{X}$
- $\mathcal{F}$ can be **any** monadic predicate you like
- $\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$ is the result of replacing **all** of the occurrences of $\mathcal{X}$ in $\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$ with $\mathcal{F}$

$\forall Z(Za \to Zb),\ Ga \vdash_2 Gb$

1	$\forall Z(Za \to Zb)$	
2	Ga	
3	$Ga \to Gb$	$\forall_2$E, 1
4	Gb	$\to$E, 3, 2

Second-Order Logic

Another Argument

Susanne is a pianist or an historian

Mary is a pianist or an historian

∴ Susanne and Mary have something in common

- This strikes me as a good argument
  - Even if Susanne isn't a historian and Mary isn't a pianist, they still have *something* in common: they are both pianists or historians!
- Unfortunately, the rules we have laid out so far will not allow us to provide a proof to vindicate this argument

Another Argument

$Ps \vee Hs$

$Pm \vee Hm$

$\therefore \exists X(Xs \wedge Xm)$

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Another Argument

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- This strikes me as a good argument
  - Even if Susanne isn't a historian and Mary isn't a pianist, they still have *something* in common: they are both pianists or historians!
- Unfortunately, the rules we have laid out so far will not allow us to provide a proof to vindicate this argument
- The trouble is that our rules only allow us to replace simple predicates, like, 'P' and 'H', with second-order variables, not complex formulas like '$Px \vee Hx$'

Comprehension

$$\mid \exists \mathcal{X} \forall \chi (\mathcal{X}\chi \leftrightarrow \mathcal{A}(\ldots \chi \ldots \chi \ldots)) \qquad \text{Comp}$$

- $\mathcal{X}$ **must not** occur in $\mathcal{A}(\ldots \chi \ldots \chi \ldots)$

Complex Properties

- Comprehension allows us to define **complex** properties
- **EXAMPLES:**
  - $\exists X \forall y(Xy \leftrightarrow (Fy \wedge Gy))$ $\Rightarrow$ the property of being F-and-G
  - $\exists X \forall y(Xy \leftrightarrow (Fy \vee Gy))$ $\Rightarrow$ the property of being F-or-G
  - $\exists X \forall y(Xy \leftrightarrow \forall Y(Yb \leftrightarrow Yy))$ $\Rightarrow$ the property of having the same properties as b
- **GENERAL PATTERN:**
  - $\exists \mathcal{X} \forall \chi(\mathcal{X}\chi \leftrightarrow \mathcal{A}(\ldots \chi \ldots \chi \ldots))$ $\Rightarrow$ the property of being $\mathcal{A}$

Notes on Comprehension

- You are allowed to plug in **any** formula for $\mathcal{A}(\ldots\chi\ldots\chi\ldots)$ (so long as it doesn't contain $\mathcal{X}$)

- *It can contain first-order quantifiers!*

$$\exists X\forall y(Xy \leftrightarrow \exists xRxy)$$

- *It can even contain second-order quantifiers!!!*

$$\exists Y\forall y(Yy \leftrightarrow \forall x\exists X(Xx \vee Xy))$$

(If we didn't allow $\mathcal{A}$ to contain second-order quantifiers, we would call it *predicative* comprehension)

$Ps \lor Hs,\ Pm \lor Hm \vdash_2 \exists X(Xs \land Xm)$

1	$Ps \lor Hs$	
2	$Pm \lor Hm$	
3	$\exists X \forall x(Xx \leftrightarrow (Px \lor Hx))$	Comp
4	$\quad \forall x(Fx \leftrightarrow (Px \lor Hx))$	
5	$\quad Fs \leftrightarrow (Ps \lor Hs)$	$\forall_1$E, 4
6	$\quad Fs$	$\leftrightarrow$E, 5, 1
7	$\quad Fm \leftrightarrow (Pm \lor Hm)$	$\forall_1$E, 4
8	$\quad Fm$	$\leftrightarrow$E, 7
9	$\quad Fs \land Fm$	$\land$I, 6, 8
10	$\quad \exists X(Xs \land Xm)$	$\exists_2$I, 9
11	$\exists X(Xs \land Xm)$	$\exists_2$E, 3, 4–10

Natural Deduction for SOL

- **Our natural deduction system for SOL:**
  - All of the natural deduction rules for FOL
  - The Introduction and Elimination rules for the second-order quantifiers
  - Comprehension

(Logicians often add another rule to this system, called *Choice*, but we will leave that one out for now)

Second-Order Logic

Set-Theory and Semantics

- The way I presented the semantics for FOL in *forallχ* was fairly informal
- Normally, philosophers use **set-theory** to formalise this semantics
- I explain this in some detail in the Primer, but I will go over a couple of the basics here
- This is important, because the **Standard Semantics** for SOL is set-theoretic
  - As you will see, not everyone agrees that the so-called "Standard Semantics" is the best semantics!

Set-Theorising our Interpretations

- In *forallχ*, I said that an interpretation specifies three things:
  - The referent of each name we are dealing with
  - The extension of each predicate we are dealing with
  - The domain of quantification

- We can think of the domain as a **set**, d

- We can also think of the extensions of our monadic predicates as **subsets** of d

$$a \subseteq b \leftrightarrow \forall x(x \in a \rightarrow x \in b)$$

The First-Order Quantifiers

- Let c be a new name added to the language
- $\forall \chi \mathcal{A}(\ldots\chi\ldots\chi\ldots)$ is true in an interpretation iff $\mathcal{A}(\ldots c\ldots c\ldots)$ is true in **every** interpretation that extends the original interpretation by assigning an object to c (without changing the interpretation in any other way)
- $\exists \chi \mathcal{A}(\ldots\chi\ldots\chi\ldots)$ is true in an interpretation iff $\mathcal{A}(\ldots c\ldots c\ldots)$ is true in **some** interpretation that extends the original interpretation by assigning an object to c (without changing the interpretation in any other way)

The Second-Order Quantifiers

- Let $\mathcal{F}$ be a new monadic predicate added to the language
- $\forall \mathcal{X}\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$ is true in an interpretation iff $\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$ is true in **every** interpretation that extends the original interpretation by assigning a subset of the domain to $\mathcal{F}$ (without changing the interpretation in any other way)
- $\exists \mathcal{X}\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$ is true in an interpretation iff $\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$ is true in **some** interpretation that extends the original interpretation by assigning a subset of the domain to $\mathcal{F}$ (without changing the interpretation in any other way)

Logical Consequence

- $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash_2 \mathcal{C}$ **iff** every interpretation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ true also makes $\mathcal{C}$ true

- Is $\vdash_2$ **sound** relative to $\vDash_2$?
  - If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash_2 \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash_2 \mathcal{C}$

- Is $\vdash_2$ **complete** relative to $\vDash_2$?
  - If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash_2 \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash_2 \mathcal{C}$

SOL is Inherently Incomplete!

- $\vdash_2$ is sound relative to $\vDash_2$, **but it is not complete!**
- This is not just because I forgot to add some rules to our proof system for SOL
- It turns out that SOL is so powerful that no system of natural deduction for SOL can be both sound and complete!
  - Proving that is well beyond our means here, since it is a corollary of Gödel's Incompleteness Theorems
  - But if you are intrigued, then I recommend you take the Foundations of Mathematics module next year
  - You won't quite learn how to prove Gödel's theorems, but you will get a sense of what they mean, and why they are so important!

Second-Order Logic

Defining Identity

- In FOL, identity is indefinable: you have to take it as a primitive, basic logical concept
- But in SOL, we can define identity!

$$\forall x \forall y (x = y \leftrightarrow_{df} \forall X(Xx \leftrightarrow Xy))$$

- **In English:** x is identical to y iff x and y have exactly the same properties

Two Leibnizian Principles

- This is an old definition of identity, which goes back to two principles proposed by Leibniz:
  - **Indiscernibility of Identicals:** $\forall x \forall y (x = y \rightarrow \forall X (Xx \leftrightarrow Xy))$
  - **Identity of Indiscernibles:** $\forall x \forall y (\forall X (Xx \leftrightarrow Xy) \rightarrow x = y)$
- The second-order definition of identity is what you get when you put these two principles together

The Identity of Indiscernibles

- You may have heard people say that while the **Indiscernibility of Identicals** is unassailable, the **Identity of Indiscernibles** is a controversial thesis
- But in fact, it is quite easy to show that the **Identity of Indiscernibles** is true on every interpretation in the Standard Semantics
- The core of the argument runs like this:
  - Suppose 'a' and 'b' refer to distinct objects, 1 and 2
  - Now consider the sentence '$Aa \leftrightarrow Ab$'
  - This will come out false if we assign $\{1\}$ to 'A' as its extension
  - So '$\forall X(Xa \leftrightarrow Xb)$' will be false

Properties and Sets

- Does this show that all of the metaphysical debate about the **Identity of Indiscernibles** was a waste of time?
- *Not at all!*
- When I introduced you to SOL, I told you that second-order quantifiers quantify over **properties**
- But in the Standard Semantics, we swapped properties for **sets**
- This swap makes a lot of sense for formal purposes, because sets are well behaved mathematical objects
- But for metaphysical purposes, it is properties which really matter

Properties and Sets

- A metaphysician should think of the Standard Semantics a a mere model of what they are really interested in
  - The sets we assign to predicates merely *represent* the properties we really care about
- When we look at the Standard Semantics like that, we must ask: How well does our set-theoretic model represent reality?
- At this point, it becomes **very** interesting to ask whether it is possible for two distinct objects to share all of their properties!

Seminar 5

- The reading for Seminar 5 is:
  - *A Second-Order Logic Primer*
- You can find this primer on the VLE
- Please attempt some the exercises
- Why not meet up in groups, and try the exercises together?