

# Rationality, Morality and Economics

## Topic 2, Lecture 2

# The Axioms of Expected Utility Theory

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# The Axioms of Expected Utility Theory

Re-Cap

Repairing the Money Pumps

Completeness

The Remaining Axioms

## The vNM Axioms

- **vNM1: Completeness**

- $A \succeq B$  or  $B \succeq A$

- **vNM2: Transitivity**

- If  $A \succeq B$  and  $B \succeq C$ , then  $A \succeq C$

- **vNM3: Independence**

- $A \succ B$  if and only if  $ApC \succ BpC$  (where  $0 < p \leq 1$ )

- **vNM4: Continuity**

- If  $A \succ B \succ C$  then there exists some  $p$  and  $q$  such that  $ApC \succ B \succ AqC$  (where  $0 < p < 1$  and  $0 < q < 1$ )

## The Money Pump Argument

- **Transitivity:** If  $A \succeq B$  and  $B \succeq C$ , then  $A \succeq C$
- Many decision theorists try to justify transitivity with a **money pump** argument
- In the standard version of a money pump argument, we show that someone with **cyclical** preferences ( $A \succ B \succ C \succ A$ ) can be tricked into paying out money for no gain
  - They start with  $C$
  - Since they prefer  $B$  to  $C$ , they are happy to pay a small fee to swap to  $B$
  - Since they prefer  $A$  to  $B$ , they are happy to pay a small fee to swap to  $A$
  - But since they prefer  $C$  to  $A$ , they are happy to pay a small fee to swap back to  $C$

## Three Objections

- (1) This money pump argument at most shows that cyclical preferences are incoherent, but there are other ways of having intransitive preferences
- (2) A sophisticated decision maker could get out of the money pump by using backward induction
- (3) Money pump arguments make the questionable assumption that the value of a sequence of choices is the sum of the values of each individual choice

## Two Objections

- (1) This money pump argument at most shows that cyclical preferences are incoherent, but there are other ways of having intransitive preferences
- (2) ~~A sophisticated decision maker could get out of the money pump by using backward induction~~
- (3) Money pump arguments make the questionable assumption that the value of a sequence of choices is the sum of the values of each individual choice

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## Two Unnecessary Features

- Johan Gustafsson (2010, 2013) pointed out that the standard money pump has two unnecessary features
  - (1) Someone **gets rich** at the expense of the agent with cyclical preferences
  - (2) The agent with cyclical preferences is presented with a **sequence** of choices
- It might seem that these features are absolutely necessary for a money pump, but we can argue against cyclical preferences without relying on them



## No One Needs To Get Rich

- Suppose Sharon has cyclical preferences
  - two sugars  $\succ$  one sugar  $\succ$  no sugar  $\succ$  two sugars
- The standard money pump argument is meant to show that I can get rich at Sharon's expense
- But why should Sharon care about that?
  - **Answer 1:** Because Sharon doesn't want people to get rich at her expense
  - **Answer 2:** Because the fact that Sharon is always willing to pay me money shows that she is never satisfied with her choice
- Either way, the crucial point isn't that I get rich at Sharon's expense; it is that Sharon *chooses against her preferences*

## A One-Off Choice

- Rather than presenting Sharon a sequence of trades, let's just offer her a *one-off choice*:
  - Sharon can choose between a tea with two sugars, a tea with one sugar, and a tea with no sugars
- Whatever Sharon chooses, she will choose against her preferences
  - Whatever choice she makes, there will be some other choice that she would prefer to have made
- Maybe this doesn't really count as a “money pump” anymore, but it is just as good a demonstration that Sharon's preferences are irrational
  - **Remember:** we already saw that the crucial point of the original money pump was just that Sharon was forced to choose against her preferences!

## The Undominated Choice Principle

- **Undominated Choice:**
  - It is irrational to choose one alternative if you prefer some other alternative
- Undominated Choice is a very intuitive principle
- But in the one-off choice we just presented to Sharon, her cyclical preferences force her to violate it
- We can construct similar one-off choices which force anyone with cyclical preferences to violate Undominated Choice

## Two Objections

- (1) This money pump argument at most shows that cyclical preferences are incoherent, but there are other ways of having intransitive preferences
- (2) ~~A sophisticated decision maker could get out of the money pump by using backward induction~~
- (3) Money pump arguments make the questionable assumption that the value of a sequence of choices is the sum of the values of each individual choice

## One Objection

- (1) This money pump argument at most shows that cyclical preferences are incoherent, but there are other ways of having intransitive preferences
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## The Varieties of Intransitive Preferences

- **Transitivity:** If  $A \succeq B$  and  $B \succeq C$ , then  $A \succeq C$
- One way of having intransitive preferences is to have cyclic preferences
  - $A \succ B \succ C \succ A$
- But here are two other ways of having intransitive preferences
  - $A \succ B \succ C \sim A$
  - $A \succ B \sim C \sim A$
- If we assume Completeness ( $A \succeq B$  or  $B \succeq A$ ) then these are the only ways of having intransitive preferences
  - We will look more closely at Completeness shortly; for now, we will take it for granted

## The Small Bonuses Approach

- **Small Bonuses:** If  $A \sim B$ , then the agent should be willing to trade  $A$  for  $B$ , if we throw in a small bonus
- Sharon's preferences: two sugars  $\succ$  one sugar  $\succ$  no sugar  $\sim$  two sugars
- We can now money pump Sharon as follows:
  - Sharon starts with a sugar-free tea
  - Sharon pays 1p to trade the sugar-free tea for a tea with one sugar
  - Sharon pays 1p to trade the tea with one sugar for a tea with two sugars
  - We pay Sharon 1p to trade the tea with two sugars for a tea with no sugar
  - Sharon's back to her original cup of tea, but has lost 1p

## The Small Bonuses Approach

- **Small Bonuses:** If  $A \sim B$ , then the agent should be willing to trade  $A$  for  $B$ , if we throw in a small bonus
- Sharon's preferences: two sugars  $\succ$  one sugar  $\sim$  no sugar  $\sim$  two sugars
- We can now money pump Sharon as follows:
  - Sharon starts with a sugar-free tea
  - Sharon pays 1p to trade the sugar-free tea for a tea with one sugar
  - We pay Sharon  $\frac{1}{3}$ p to trade the tea with one sugar for a tea with two sugars
  - We pay Sharon  $\frac{1}{3}$ p to trade the tea with two sugars for a tea with no sugar
  - Sharon's back to her original cup of tea, but has lost  $\frac{1}{3}$ p



## The Small Bonuses Approach

- **Small Bonuses:** If  $A \sim B$ , then the agent should be willing to trade  $A$  for  $B$ , if we throw in a small bonus
- Sharon's preferences: two sugars  $\succ$  one sugar  $\sim$  no sugar  $\sim$  two sugars
- We can now money pump Sharon as follows:
  - Sharon starts with a sugar-free tea
  - Sharon pays 3p to trade the sugar-free tea for a tea with one sugar
  - We pay Sharon 1p to trade the tea with one sugar for a tea with two sugars
  - We pay Sharon 1p to trade the tea with two sugars for a tea with no sugar
  - Sharon's back to her original cup of tea, but has lost 1p

## A Problem for the Small Bonuses Approach

- Why think that the Small Bonuses principle is true?
  - **Small Bonuses:** If  $A \sim B$ , then the agent should be willing to trade  $A$  for  $B$ , if we throw in a small bonus
- We might think that we can support Small Bonuses with the following argument:

$$A \sim B$$

$$B + \epsilon \succ B$$

$$\therefore B + \epsilon \succ A$$

- But this argument presupposes a version of the transitivity principle for preference, and that is what we are trying to argue for!

## The Lottery Approach

- Johan Gustafsson (2010) has suggested a different way of trying to extend the money pump argument to cover all intransitive preferences
- **Dominance for Lotteries**
  - $L_1 \succ L_2$  if: there is at least one state where the outcome of  $L_1$  is strictly preferred to the outcome of  $L_2$ , and no state where the outcome of  $L_2$  is strictly preferred to the outcome of  $L_1$
- We can use this principle to transform *intransitive* preferences into *cyclic* preferences over lotteries

## The Lottery Approach

- $A \succ B \succ C \sim A$

	$S_1$	$S_2$	$S_3$
$L_1$	$A$	$B$	$C$
$L_2$	$B$	$C$	$A$
$L_3$	$C$	$A$	$B$

- $L_1 \succ L_2 \succ L_3 \succ L_1$
- We already know that a successful version of the money pump argument can be used on cyclic preferences

## The Lottery Approach

- $A \succ B \sim C \sim A$

	$S_1$	$S_2$	$S_3$
$L_1$	$A$	$B$	$C$
$L_2$	$B$	$C$	$A$
$L_3$	$C$	$A$	$B$

- $L_1 \succ L_2 \succ L_3 \succ L_1$
- We already know that a successful version of the money pump argument can be used on cyclic preferences

## One Objection

- (1) This money pump argument at most shows that cyclical preferences are incoherent, but there are other ways of having intransitive preferences
- (2) ~~A sophisticated decision maker could get out of the money pump by using backward induction~~
- (3) ~~Money pump arguments make the questionable assumption that the value of a sequence of choices is the sum of the values of each individual choice~~

## No Objections!

- (1) ~~This money pump argument at most shows that cyclical preferences are incoherent, but there are other ways of having intransitive preferences~~
- (2) ~~A sophisticated decision maker could get out of the money pump by using backward induction~~
- (3) ~~Money pump arguments make the questionable assumption that the value of a sequence of choices is the sum of the values of each individual choice~~

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## Two Questions about Completeness

- **Completeness:**  $A \succeq B$  or  $B \succeq A$
- **Two Questions**
  - (1) Is Completeness true of real people like you and me?
  - (2) Would Completeness be true of an ideally rational agent?

## Preference and Choice

- (1) Is Completeness true of real people like you and me?
- **A Proposal:**
    - To prefer  $A$  to  $B$  is to be disposed to choose  $A$  over  $B$
    - To prefer  $B$  to  $A$  is to be disposed to choose  $B$  over  $A$
    - To be indifferent between  $A$  and  $B$  is to lack a disposition to choose one over the other
  - If this proposal is right, then you can't fail to satisfy Completeness
  - However that is a big **'if'**

## Is Completeness Rationally Required?

(2) Would Completeness be true of an ideally rational agent?

- It certainly isn't immediately obvious that it would
  - It does not seem like people who are stumped by stupid 'Would you rather...?' questions are being *irrational*
- And in fact, there is a famous argument *against* the idea that Completeness is rationally required of us, known as the **Small Improvement Argument**

## The Small Improvement Argument

*“Suppose you must determine which of a cup of coffee and a cup of tea tastes better to you. The coffee has a full-bodied, sharp, pungent taste, and the tea has a warm, soothing, fragrant taste. It is surely possible that you rationally judge that the cup of Sumatra Gold tastes neither better nor worse than the cup of Pearl Jasmine, and that although a slightly more fragrant cup of the Jasmine would taste better than the original, the more fragrant Jasmine would not taste better than the cup of coffee.”*

*(Ruth Chang, 2002, p. 669)*

## The Small Improvement Argument

- (i) tea  $\not\sim$  coffee and coffee  $\not\sim$  tea
- $\therefore$  (ii) if Completeness is true, tea  $\sim$  coffee
- (iii) tea<sup>+</sup>  $\succ$  tea
- $\therefore$  (iv) if Completeness is true, tea<sup>+</sup>  $\succ$  coffee
- (v) but tea<sup>+</sup>  $\not\sim$  coffee
- $\therefore$  (vi) Completeness is not true

### PROBLEM

- The step from (ii) and (iii) to (iv) is an application of Transitivity
- But our argument for Transitivity as a rational requirement assumed Completeness!

## The Small Improvement Argument

- (i) tea  $\not\sim$  coffee and coffee  $\not\sim$  tea
- $\therefore$  (ii) if Completeness is true, tea  $\sim$  coffee
- (iii) tea<sup>+</sup>  $\succ$  tea
- $\therefore$  (iv) if Completeness is true, tea<sup>+</sup>  $\succ$  coffee
- (v) but tea<sup>+</sup>  $\not\sim$  coffee
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### SOLUTION

- If Completeness were a rational requirement, then Transitivity would be a rational requirement
- The Small Improvement Argument shows that if Transitivity is a rational requirement, then Completeness is not
- So Completeness is rationally unstable: if it were a rational requirement, then it would not be

## Broome's Argument for Completeness

*“Suppose two careers are open to you: a career in the army and a good career as a priest. Suppose they are incommensurate in their goodness. Then choosing either would not be wrong. You have to choose without the guidance of reason, and suppose you choose the army: you commit yourself to the army career, and give up the chance of a good career in the church. In doing so, you are doing nothing wrong.”*

## Broome's Argument for Completeness

*But then suppose another opportunity comes up to join the church, this time in much worse conditions. You now face a choice between the army or a much less good career as a priest. Suppose these two, also, are incommensurate. Choosing either would not be wrong. You have to choose without the guidance of reason. Suppose this time you choose the church. Once again, you do nothing wrong.*

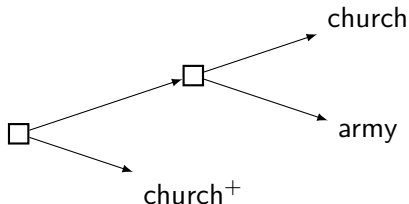


## Broome's Argument for Completeness

*But though you have not acted wrongly in either of your choices, the effect of the two together is that you end up with a much worse career in the church than you could have had."*

*(John Broome, 1999, p. 156)*

## Broome's Argument for Completeness



- On the one hand, you seem permitted to go either way at each choice point
  - At each choice point, you act “without the guidance of reason”
- But on the other hand, the whole sequence of choices which ends with the worse church career seems irrational
  - You could have gotten an outcome you preferred more!

## Broome's Argument for Completeness

- Broome's argument for Completeness is obviously similar to the money pump arguments for Transitivity
- But it is important to note the following difference
  - The money pump arguments aim to show that cyclical preferences **force** you to act against your preferences
  - Broome's only shows that incomplete preferences **allow** you to act against your preferences
- How much weight you think Broome's argument carries will therefore depend on whether you think that rationality should serve as a surefire guarantee that you will not act against your preferences

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- $A \succ B$  if and only if  $ApC \succ BpC$  (where  $0 < p \leq 1$ )

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- If  $A \succ B \succ C$  then there exists some  $p$  and  $q$  such that  $ApC \succ B \succ AqC$  (where  $0 < p < 1$  and  $0 < q < 1$ )

## Independence

- **vNM3: Independence**

- $A \succ B$  if and only if  $ApC \succ BpC$  (where  $0 < p \leq 1$ )

- Some people have proposed money-pump-style arguments in favour of this axiom (see Peterson, §8.4)
- However, it also generates Allais' Paradox (see Peterson §4.4)

## Continuity

- **vNM4: Continuity**
  - If  $A \succ B \succ C$  then there exists some  $p$  and  $q$  such that  $ApC \succ B \succ AqC$  (where  $0 < p < 1$  and  $0 < q < 1$ )
- People offer little by way of argument for this principle
- It's usually described as an “uncontroversial technical assumption”

## For the Seminar

- Please read
  - Peterson, *An Introduction Decision Theory*, Chapter 8: Bayesianism and Pragmatic Arguments
  - Rabinowicz, 'Safeguards of a Disunified Mind'
- Please see the VLE page for instructions on how to prepare for the seminar



## References

- Broome, John (1999) *Ethics out of Economics*, Cambridge: Cambridge University Press
- Chang, Ruth (2002) 'The Possibility of Parity', *Ethics* 112: 659–688
- Gustafsson, Johan (2010) 'A Money-Pump for Acyclic Intransitive Preferences', *Dialectica* 64: 251–257
- — (2013) 'The Irrelevance of the Diachronic Money-Pump Argument for Acyclicity', *The Journal of Philosophy* 110: 460–464