Intermediate Logic Spring
Lecture Nine

Fitch's Paradox

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Fitch's Paradox

Assertibility Conditions

- Last week we looked at one of the standard semantics for IL
- The guiding idea is that TRUTH is not the fundamental semantic concept; WARRANTED ASSERTIBILITY is
  - Once you have told me the conditions in which I would be warranted to assert $\mathcal{A}$, you have told me everything about what $\mathcal{A}$ means
- In mathematical discourse, a sentence is assertible iff it is **provable**
- The BHK semantics gave us a way of describing what a proof of a (complex) sentence would consist in

The BHK Semantics

(1) A proof of $\mathcal{A} \land \mathcal{B}$ consists of a proof of $\mathcal{A}$ is and a proof of $\mathcal{B}$

(2) A proof of $\mathcal{A} \lor \mathcal{B}$ consists of a proof of $\mathcal{A}$ or a proof of $\mathcal{B}$

(3) A proof of $\mathcal{A} \to \mathcal{B}$ consists of a method for converting any proof of $\mathcal{A}$ into a proof of $\mathcal{B}$

(4) A proof of $\mathcal{A} \leftrightarrow \mathcal{B}$ consists of a proof of $\mathcal{A} \to \mathcal{B}$ and a proof of $\mathcal{B} \to \mathcal{A}$

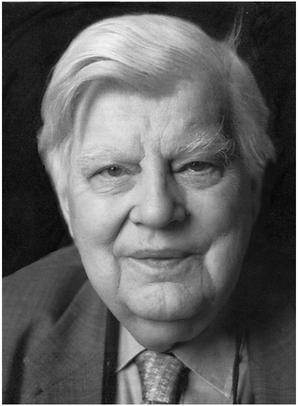
(5) A proof of $\neg\mathcal{A}$ consists of a proof of $\mathcal{A} \to \bot$

(6) A proof of $\exists \chi \mathcal{A}(\chi)$ consists of a proof of $\mathcal{A}(c)$, for some element of the domain, c

(7) A proof of $\forall \chi \mathcal{A}(\chi)$ consists of a method which acts on any element in the domain, c, and delivers a proof that $\mathcal{A}(c)$

Dummett's Semantic Arugments

- Dummett used the BHK semantics to argue for IL
- Dummett presented two arguments
  - The Manifestation Argument
  - The Acquisition Argument
- We focussed on the Manifestation Argument



Michael Dummett

Dummett's Manifestation Argument

- If TRUTH were the fundamental semantic concept, then to understand a sentence would be to know its truth-conditions
- Whatever exactly our understanding of a sentence consists in, that understanding must be *manifestable*
- But there would be no way of manifesting knowledge of the truth-conditions of **undecidable** mathematical sentences
- So TRUTH cannot be the fundamental semantic concept
- We should replace it in mathematical contexts with PROVABILITY, since our knowledge of what it takes to prove a sentence is manifestable

Generalising: Dummettian Anti-Realism

- So far we have focussed on mathematical discourse, because it has fairly clear rules on assertion: assertibility = provability
- But as Dummett was well aware, if his argument works for mathematical discourse, a version of it should work elsewhere too
- In general, Dummettian considerations cast doubt on the whole idea of **verification-transcendent truth-conditions**
  - A sentence's truth-conditions are *verification-transcendent* iff it exceeds our ability to verify or falsify whether those conditions are satisfied
- How would we ever manifest knowledge of verification-transcendent truth-conditions!?

All Truths are Knowable

- This Dummettian line of thought seems to motivate to the following principle:
  - **Knowability:** All truths are knowable
- This Knowability Principle should be thought of as describing an *in principle* kind of possibility
  - It may be that some truths are so complex that no real life human could ever know them
  - Every truth is *in principle* knowable, if only by a super-being with a much more powerful mind than any human's

Knowability and Verificationism

- This Knowability Principle is a pared down version of **verificationism**
- The classical verificationists thought that every meaningful sentence could be verified or falsified in terms of *sense-data*
- Knowability doesn't mention sense-data; it is just says that the totality of truths does not outstrip what could in principle be known
- Any philosopher who has ever felt pulled towards any version of verificationism will be attracted to Knowability

All Truths are *Known*!?

- Unfortunately, an argument known as **Fitch's Paradox** takes the not-obviously-silly Knowability Principle, and turns it into something absurd:
  - **Omniscience:** All truths are known
- The Omniscience Principle is ridiculous: there are plenty of truths that no one has known or ever will know
  - How many hairs did Julius Caesar have on his head the day he died?
- So if Fitch's Paradox works then the Knowability Principle, which lies behind Dummettian intuitionism, must be false!

Fitch's Paradox

Formalising the Knowability Principle

- We can combine Modal Logic and Second-Order Logic to symbolise Knowability
  - **Knowability:** All truths are knowable
  - **In symbols:** $\forall P(P \rightarrow \Diamond KP)$
- KP means that P is, was or will be known at some time or other; so $\Diamond KP$ means that it is possible for P to be known at some time or other
- The quantifier binds a variable in **sentence** position; this is a kind of *second-order* variable

Quantification into Sentence Position

- When we studied SOL, we kept things simple and assumed that every second-order variable has **one** argument place
  - **In other words:** we assumed that every second-order variable combines with *one* term to make a sentence
- But we can let second-order variables have any number of places that we like
  - **Example:** A two-place second-order variable combines with *two* terms to make a sentence
- We can even use 0-place second-order variables
  - A 0-place second-order variable is a variable which does not need to be combined with any terms to make a sentence
  - **In other words:** a 0-place second-order variable is a variable which replaces *whole sentences*

Quantification over Propositions

- Unsurprisingly, there is a controversy about how to read quantification into sentence position!
- However, it can be helpful to read it as quantification over propositions
  - $\forall P(P \vee \neg P)$ $\Rightarrow$ Every proposition is either true or false
  - $\exists P(\neg P)$ $\Rightarrow$ Some proposition is false
  - **Knowability:** $\forall P(P \rightarrow \Diamond KP)$ $\Rightarrow$ Every true proposition is knowable

Formalising Fitch's Paradox

- We can also use quantification into sentence position to formalise the Omniscience Principle
  - **Omniscience:** All truths are known
  - **In symbols:** $\forall P(P \rightarrow KP)$
- Fitch's Paradox consists of a proof vindicating the following argument:

$$\forall P(P \rightarrow \Diamond KP) \ \therefore \ \forall P(P \rightarrow KP)$$

- This argument uses the familiar rules for Modal Logic and SOL, but also adds a couple of plausible rules governing *knowledge*

Factivity

n	$K\mathcal{A}$	
	$\mathcal{A}$	Factivity, n

- The Factivity Rule is meant to capture the idea that you cannot *know* anything which isn't *true*

K-Distribution

n	$K(\mathcal{A} \wedge \mathcal{B})$	
	$K\mathcal{A} \wedge K\mathcal{B}$	K-Dist, n

- The K-Distribution Rule is meant to capture the idea that you cannot know a *conjunction* without knowing each *conjunct*

$\forall P(P \to \Diamond KP) \therefore \forall P(P \to KP)$

1	$\forall P(P \to \Diamond KP)$	
2	$\quad A$	
3	$\quad\quad \neg KA$	
4	$\quad\quad A \wedge \neg KA$	$\wedge$I, 2, 3
5	$\quad\quad (A \wedge \neg KA) \to \Diamond K(A \wedge \neg KA)$	$\forall_2$E, 1
6	$\quad\quad \Diamond K(A \wedge \neg KA)$	$\to$E, 5, 4

1	$\vert\ \vert\ $	
2	$\vert\ \vert\ \vert\ K(A \wedge \neg KA)$	
3	$\vert\ \vert\ \vert\ KA \wedge K\neg KA$	K-Dist, 2
4	$\vert\ \vert\ \vert\ KA$	$\wedge$E, 3
5	$\vert\ \vert\ \vert\ K\neg KA$	$\wedge$E, 3
6	$\vert\ \vert\ \vert\ \neg KA$	Factivity, 5
7	$\vert\ \vert\ \vert\ \bot$	$\bot$I, 4, 6
8	$\vert\ \vert\ \neg K(A \wedge \neg KA)$	$\neg$I, 2–7
9	$\vert\ \Box\neg K(A \wedge \neg KA)$	Nec, 1–8

1	$\forall P(P \to \Diamond KP)$	
2	A	
3	$\neg KA$	
4	$A \wedge \neg KA$	$\wedge$I, 2, 3
5	$(A \wedge \neg KA) \to \Diamond K(A \wedge \neg KA)$	$\forall_2$E, 1
6	$\Diamond K(A \wedge \neg KA)$	$\to$E, 5, 4
7	$\Box\neg K(A \wedge \neg KA)$	Other Proof
8	$\neg\Diamond K(A \wedge \neg KA)$	MC, 7
9	$\bot$	$\bot$I, 6, 8
10	$\neg\neg KA$	$\neg$I, 3–9
11	KA	DNE, 10
12	$A \to KA$	$\to$I, 2–11
13	$\forall P(P \to KP)$	$\forall_2$I, 12

Bad Response One

- There are good philosophical reasons to think that every truth is knowable
- Fitch's Paradox shows that this entails that every truth is known
- So we should just accept that every truth is known

- This is a **bad response** because it is obviously absurd to say that every truth is known!
  - That may be a little bit strong — if you believe in God you might be happy to say that every truth is known
  - But do we really want such a neat and tidy proof that God exists!?

Bad Response Two

- It is obviously absurd to say that every truth is known
- Fitch's Paradox shows that this is entailed by the claim that every truth is knowable
- So we should deny that every truth is knowable

- This also seems like a **bad response**, because the Knowability Principle seems genuinely plausible and interesting!
  - It may turn out that Knowability is false, but if so, that will be a substantive philosophical discovery
  - Knowability should not be rejected as plain silly

How Should We Deal With Fitch's Paradox?

- If you agree that both of these responses are **bad responses**, then there is only one option left
- We need to find some sort of error in the reasoning used in Fitch's Paradox
- And as it happens, intuitionists have a suggestion about what that error might be...

Fitch's Paradox

The Intuitionistic Response

- Fitch's Paradox is meant to refute Knowability, which in turn is meant to undermine intuitionism
- But some intuitionists reply by pointing out that Fitch's Paradox is not **intuitionistically valid**
- As a result, it does nothing to undermine intuitionism
  - Williamson (1982) 'Intuitionism Disproved?'
  - Dummett (2009) 'Fitch's Paradox of Knowability'

Timothy Williamson

1	$\forall P(P \to \Diamond KP)$	
2	$\quad A$	
3	$\quad\quad \neg KA$	
4	$\quad\quad A \wedge \neg KA$	$\wedge$I, 2, 3
5	$\quad\quad (A \wedge \neg KA) \to \Diamond K(A \wedge \neg KA)$	$\forall_2$E, 1
6	$\quad\quad \Diamond K(A \wedge \neg KA)$	$\to$E, 5, 4
7	$\quad\quad \Box\neg K(A \wedge \neg KA)$	Other Proof
8	$\quad\quad \neg\Diamond K(A \wedge \neg KA)$	MC, 7
9	$\quad\quad \bot$	$\bot$I, 6, 8
10	$\quad \neg\neg KA$	$\neg$I, 3–9
11	$\quad KA$	DNE, 10
12	$A \to KA$	$\to$I, 2–11
13	$\forall P(P \to KP)$	$\forall_2$I, 12

1	$\forall P(P \to \Diamond KP)$	
2	$\quad A$	
3	$\quad\quad \neg KA$	
4	$\quad\quad A \wedge \neg KA$	$\wedge$I, 2, 3
5	$\quad\quad (A \wedge \neg KA) \to \Diamond K(A \wedge \neg KA)$	$\forall_2$E, 1
6	$\quad\quad \Diamond K(A \wedge \neg KA)$	$\to$E, 5, 4
7	$\quad\quad \Box \neg K(A \wedge \neg KA)$	Other Proof
8	$\quad\quad \neg \Diamond K(A \wedge \neg KA)$	MC, 7
9	$\quad\quad \bot$	$\bot$I, 6, 8
10	$\quad \neg\neg KA$	$\neg$I, 3–9
11	$\quad KA$	DNE, 10
12	$A \to KA$	$\to$I, 2–11
13	$\forall P(P \to KP)$	$\forall_2$I, 12

1	$\forall P(P \to \Diamond KP)$	
2	$\quad A$	
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4	$\quad\quad A \wedge \neg KA$	$\wedge$I, 2, 3
5	$\quad\quad (A \wedge \neg KA) \to \Diamond K(A \wedge \neg KA)$	$\forall_2$E, 1
6	$\quad\quad \Diamond K(A \wedge \neg KA)$	$\to$E, 5, 4
7	$\quad\quad \Box \neg K(A \wedge \neg KA)$	Other Proof
8	$\quad\quad \neg \Diamond K(A \wedge \neg KA)$	MC, 7
9	$\quad\quad \bot$	$\bot$I, 6, 8
10	$\quad \neg\neg KA$	$\neg$I, 3–9
11	$A \to \neg\neg KA$	$\to$I, 2–10
12	$\forall P(P \to \neg\neg KP)$	$\forall_2$I, 11

How Is This Any Better?

- In IL, $\forall P(P \rightarrow \Diamond KP)$ **does not** imply $\forall P(P \rightarrow KP)$
- In IL, $\forall P(P \rightarrow \Diamond KP)$ **only** implies $\forall P(P \rightarrow \neg\neg KP)$
- *But how is that any better!?*

  $\forall P(P \rightarrow KP) \Rightarrow$ All truths are known

  $\forall P(P \rightarrow \neg\neg KP) \Rightarrow$ All truths are not not known

- The important thing to remember is that in IL, 'not' doesn't mean quite the same thing as it does in Classical Logic

BHK on Negation

- According to BHK, a proof of $\neg\mathcal{A}$ consists of a proof of $\mathcal{A} \rightarrow \bot$
- And according to BHK, a proof of $\mathcal{A} \rightarrow \mathcal{B}$ consists of a method for converting any proof of $\mathcal{A}$ into a proof of $\mathcal{B}$
- So according to BHK, a proof of $\neg\mathcal{A}$ consists of a method of converting a proof of $\mathcal{A}$ into a proof of $\bot$

What Intuitionistic Negation Means

- For an intuitionist, $\neg\mathcal{A}$ means that there is a procedure for converting a proof of $\mathcal{A}$ into a proof of $\bot$
- **A little more roughly:** For an intuitionist, $\neg\mathcal{A}$ means that it is *impossible* to prove $\mathcal{A}$
- These rough glosses only work when we are focussing on contexts where assertibility = provability
- **More generally:** For an intuitionist, $\neg\mathcal{A}$ means that it is impossible to have warrant to assert $\mathcal{A}$

Not Not Knowing

- For an intuitionist, $\neg\neg\mathcal{A}$ means that it is impossible to have warrant to assert that it is impossible to have warrant to assert $\mathcal{A}$
- **More simply put:** For an intuitionist, $\neg\neg\mathcal{A}$ means that it is impossible to have warrant to deny $\mathcal{A}$
- So for an intuitionist, $\neg\neg K\mathcal{A}$ means that it is impossible to have warrant to deny that it was or will ever be known that $\mathcal{A}$

Back to Fitch's Paradox

- In IL, $\forall P(P \rightarrow \Diamond KP)$ only implies $\forall P(P \rightarrow \neg\neg KP)$
- $\forall P(P \rightarrow \neg\neg KP) \Rightarrow$ For any true proposition P, it is impossible to have warrant to deny that P was ever or will ever be known
- That principle no longer sounds absurd
- In fact, Dummett even suggests that $\forall P(P \rightarrow \neg\neg KP)$ is a better formalisation of the Knowability Principle than $\forall P(P \rightarrow \Diamond KP)$!

Fitch's Paradox

An Objection to the Intuitionistic Response

- The intuitionistic response is very neat and clever, but not everyone is convinced
- In particular, Neil Tennant — who is an intuitionist himself — does not think that the response works
  - See the very beginning of his 2002 paper, 'Victor Vanquished'

Neil Tennant

$\forall P(P \to \neg\neg KP) \vdash_I \forall P(\neg KP \to \neg P)$

1	$\forall P(P \to \neg\neg KP)$	
2	$\quad \neg KA$	
3	$\quad A \to \neg\neg KA$	$\forall_2$E, 1
4	$\quad\quad \neg\neg KA$	
5	$\quad\quad \bot$	$\bot$I, 2, 4
6	$\quad \neg\neg\neg KA$	$\neg$I, 4–5
7	$\quad \neg A$	MT, 3, 6
8	$\neg KA \to \neg A$	$\to$I, 2–7
9	$\forall P(\neg KP \to \neg P)$	$\forall_2$I, 8

All Unknown Propositions are False

- If intuitionists accept $\forall P(P \rightarrow \neg\neg KP)$, then they have to accept $\forall P(\neg KP \rightarrow \neg P)$
- $\forall P(\neg KP \rightarrow \neg P) \Rightarrow$ Any proposition which is not known to be true (at some time or other) is false
- Understood like that, this principle sounds pretty absurd!

Another Intuitionistic Response

- For an intuitionist, $\neg\mathcal{A}$ means that it is impossible to have warrant to assert $\mathcal{A}$
- For an intuitionist, $\neg K\mathcal{A}$ means that it is impossible to have warrant to assert that it was or will ever be known that $\mathcal{A}$
- $\forall P(\neg KP \to \neg P) \Rightarrow$ For any proposition P, if it is impossible to have warrant to assert that P is known to be true, then it is impossible to have warrant to assert that P
- That no longer looks absurd — or at least it is not *obviously* absurd!

Fitch's Paradox

Autumn Term: Classical Logic

- In the Autumn Term, you learned how to use classical TFL and FOL
- These are extremely important tools in analytic philosophy!
- Even if you do not normally draw up natural deduction proofs in a paper, understanding how these proof systems work will certainly help you think through arguments more carefully

Spring Term: Non-Classical Logics

- In the Spring Term, we looked at three non-classical logics
  - Modal Logic
  - Second-Order Logic
  - Intuitionistic Logic
- Part of the reason for studying these logics is, again, that they are useful tools in analytic philosophy
- But even more importantly, these logics are *themselves* philosophically interesting
- Above all, I hope that this term has shown you that studying logic isn't *just* a way of helping you to study philosophy
- Studying logic is *itself* a way of studying philosophy!

Tomorrow's Seminar

- For tomorrow's seminar, please read:
  - Timothy Williamson, 'Intuitionism Disproved?'
  - Dorothy Edgington, 'The Paradox of Knowability'
  - Timothy Williamson, 'on the Paradox of Knowability'
- All three of these articles are very short, and they are all available via the Reading List on the VLE
- I have posted some study questions on the VLE to go with these readings
- It would also be very helpful if you tried to come up with a question or two of your own, and brought those questions in writing