Intermediate Logic Spring
Lecture Eight

The BHK Semantics

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The BHK Semantics

Restricting Classical Logic

- IL is a **restriction** of classical FOL
- The natural deduction system for IL includes all of the basic rules for FOL, **apart from TND**

i	$\mathcal{A}$	
j	$\mathcal{B}$	
k	$\neg\mathcal{A}$	
l	$\mathcal{B}$	
	$\mathcal{B}$	TND, i–j, k–l

Rejecting Derived Rules of FOL

- If we reject the basic rule of TND, then we have to reject a number of derived rules too

- Most obviously, we have to reject DNE:

m	$\neg\neg\mathcal{A}$	
	$\mathcal{A}$	DNE, m

Rejecting Derived Rules of FOL

- We also have to reject one of the De Morgan Rules

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

- But we get to keep the other three De Morgan Rules!

Rejecting Derived Rules of FOL

- We also have to reject one of the rules for Converting Quantifiers

m	$\neg\forall x\mathcal{A}$	
	$\exists x\neg\mathcal{A}$	CQ, m

- But we get to keep the other three rules for Converting Quantifiers!

Arguing for Intuitionism

- Last week we looked at a **proof-theoretic** argument for IL
- This week we will look at the **semantics** for IL
- We will then look at an argument for IL developed by Michael Dummett
- According to this argument, we should prefer the semantics for IL over the classical semantics (at least for mathematical discourse), and so should accept IL

The BHK Semantics

LEM and Bivalence

- Last week, we saw that intuitionists reject the **Law of the Excluded Middle** (LEM): $\mathcal{A} \vee \neg\mathcal{A}$
  - Importantly, they **do not** assert any instance of $\neg(\mathcal{A} \vee \neg\mathcal{A})$
  - $\neg(\mathcal{A} \vee \neg\mathcal{A})$ is an intuitionistic contradiction!

- Intuitively, there seems to be a close link between LEM and a the semantic principle of *Bivalence*
  - **Bivalence:** Every sentence is either true or false (but not both)

- Bivalence guarantees that every instance of LEM will be true
  - If $\mathcal{A}$ true, then $\mathcal{A} \vee \neg\mathcal{A}$ will obviously be true too
  - If $\mathcal{A}$ is false, then $\neg\mathcal{A}$ will be true, and so $\mathcal{A} \vee \neg\mathcal{A}$ will still be true

Adding an Extra Truth-Value

- **A natural thought:** we could cook up a semantics for IL if we rejected Bivalence, and added another truth-value
- Imagine we said that every sentence has one of three truth-values: *True* (T), *False* (F) or *Neither* (N)
- We could then say that some instances of $\mathcal{A} \vee \neg\mathcal{A}$ are neither true nor false:
  - Imagine the truth-value of $\mathcal{A}$ is N
  - In that case, it would be natural to say that the truth-value of $\neg\mathcal{A}$ is N too
  - And then it would be natural to conclude that the truth-value of $\mathcal{A} \vee \neg\mathcal{A}$ is also N

It Won't Work!

- As natural as all this is, Gödel proved that it wouldn't work, given these two assumptions:
  - (i) A disjunction is true iff at least one of its disjuncts is true
  - (ii) $\mathcal{A} \leftrightarrow \mathcal{B}$ is true iff $\mathcal{A}$ and $\mathcal{B}$ have the same truth-value
- Given these assumptions, this disjunction has to be true:

$$(A \leftrightarrow B) \vee (A \leftrightarrow C) \vee (A \leftrightarrow D) \vee (B \leftrightarrow C) \vee (B \leftrightarrow D) \vee (C \leftrightarrow D)$$

- There are *four* atomic sentences, but only *three* truth-values; so two sentences must have the same truth-value; so by (ii) one of the disjuncts must be true; so by (i) the disjunction must be true
- However, there is no way of *proving* this disjunction in IL

From Heyting to Gödel

It is as if you had a malicious pleasure in showing the purposelessness of others' investigations [...] In the sense of economy of thought this work is certainly useful, and in addition to that comes the particular beauty of your short proof.

(Letter from Heyting to Gödel, 1932)

An Intuitionist's Options

- **General Result:** Given (i) and (ii), it is impossible to construct a semantics for IL which has finitely many truth-values
  (i) A disjunction is true iff at least one of its disjuncts is true
  (ii) $\mathcal{A} \leftrightarrow \mathcal{B}$ is true iff $\mathcal{A}$ and $\mathcal{B}$ have the same truth-value

- **Options:**
  (a) Try adding *infinitely* many truth-values
  (b) Try rejecting (i) or (ii)
  (c) Try fundamentally changing our whole approach to semantics

- In this lecture, we are going to pursue option (c)

The BHK Semantics

What is the Fundamental Semantic Concept?

- Every semantic theory we have looked at so far has taken TRUTH to be the fundamental semantic concept

- **Guiding Idea:** To understand a sentence is to know its truth-conditions

- The semantic theories gave us a way of displaying the truth-conditions of the sentences in the language at hand
  - $\mathcal{A} \wedge \mathcal{B}$ is **true** iff $\mathcal{A}$ is true and $\mathcal{B}$ is true
  - $\Box\mathcal{A}$ is **true** iff $\mathcal{A}$ is true at every world
  - $\forall \chi \mathcal{A}(\chi)$ is **true** iff $\mathcal{A}(c)$ is true, no matter what object in the domain is named by c
  - $\forall \mathcal{X} \mathcal{A}(\mathcal{X})$ is **true** iff $\mathcal{A}(\mathcal{F})$ is true, no matter what subset of the domain is the extension of $\mathcal{F}$

What is the Fundamental Semantic Concept?

- Many intuitionists reject the assumption that TRUTH is the fundamental semantic concept
- They take WARRANTED ASSERTIBILITY to the be the fundamental semantic concept instead
- **Guiding Idea:** To understand a sentence is to know under what circumstances you would be warranted to assert it
- Once you have told me when I would be warranted to assert a sentence, you have told me everything there is to know about what that sentence means

Two Different Perspectives on Language

- **The fundamental semantic concept is** TRUTH
  - When we take this approach, we are thinking of language as fundamentally *representational*
  - The distinctive thing about language is that sentences *represent* the world, truly or falsely

- **The fundamental semantic concept is** WARRANTED ASSERTIBILITY
  - When we take this approach, we are thinking of language as fundamentally something we *use* for various purposes
  - Our semantic theory needs to tell us the rules for how to use a given sentence

When are You Warranted to Assert a Sentence?

- **Question:** What does it mean to say that we are warranted to assert a given sentence?
- There is probably no single answer to this question; different areas of discourse seem to be governed by different rules for assertion
  - In scientific contexts, you cannot assert a sentence unless you have good experimental (or maybe theoretical) evidence in its favour
  - In everyday contexts, you are allowed to assert things for which you have no more backing than your own memory

From Assertibility to Provability

- There is one area of discourse where the rules of assertion are pretty clear
  - In **mathematics**, you cannot assert a sentence unless you have a *proof* for it

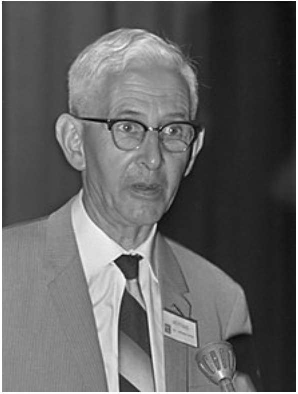
- So for intuitionists, if we are presenting a semantics for a mathematical language, we should take PROOF to be our fundamental notion

- Once you have told me what it would take to *prove* a mathematical sentence, you have told me everything there is to know about what that sentence means

  (**Remember:** Intuitionism started life as a philosophy of mathematics)

Introducing the BHK Semantics

- This kind of semantics was rigorously developed by Heyting, and independently by Kolmogorov
- It is now known as the BHK semantics
  - The extra 'B' is for Brouwer, the inventor of intuitionism
- The semantics gives us a method of describing proofs of more complex sentences in terms of proofs of simpler sentences



Arend Heyting

The BHK Semantics

(1) A proof of $\mathcal{A} \wedge \mathcal{B}$ consists of a proof of $\mathcal{A}$ is and a proof of $\mathcal{B}$

(2) A proof of $\mathcal{A} \vee \mathcal{B}$ consists of a proof of $\mathcal{A}$ or a proof of $\mathcal{B}$

(3) A proof of $\mathcal{A} \rightarrow \mathcal{B}$ consists of a method for converting any proof of $\mathcal{A}$ into a proof of $\mathcal{B}$

(4) A proof of $\mathcal{A} \leftrightarrow \mathcal{B}$ consists of a proof of $\mathcal{A} \rightarrow \mathcal{B}$ and a proof of $\mathcal{B} \rightarrow \mathcal{A}$

(5) A proof of $\neg\mathcal{A}$ consists of a proof of $\mathcal{A} \rightarrow \bot$

(6) A proof of $\exists \chi \mathcal{A}(\chi)$ consists of a proof of $\mathcal{A}(c)$, for some element of the domain, c

(7) A proof of $\forall \chi \mathcal{A}(\chi)$ consists of a method which acts on any element in the domain, c, and delivers a proof that $\mathcal{A}(c)$

The BHK Semantics

BHK and LEM

- Let's take G to be a statement of *Goldbach's Conjecture*
  - **Goldbach's Conjecture:** Every even number greater than 2 is the sum of two primes
- As things stand, no one has a proof of G, but nor do they have a proof of $\neg G$
- What is more, this may not be because we haven't been smart enough: it may just be that there is no proof of G or of $\neg G$
- If so then, according to the BHK semantics, there would be no proof of $G \vee \neg G$
- So $G \vee \neg G$ would not be assertible

An Instance of Goldbach's Conjecture

- Consider the following *instance* of Goldbach's Conjecture:
  - If 2^{100} is an even number greater than 2, then it is the sum of two primes
- 2^{100} is an absolutely huge even number, and I have no idea if anyone has ever checked whether it is the sum of two primes
- But I know that we **could** check if we liked
  - Just go through all of the primes smaller than 2^{100}, and see if any pair of them add up to 2^{100}
- **Technical Terminology:** It is *decidable* whether 2^{100} is the sum of two primes
  - We have a finite procedure for proving or refuting the claim that 2^{100} is the sum of two primes

An Instance of Goldbach's Conjecture

- Since it is decidable whether 2^{100} is the sum of two primes, the BHK semantics tells us that this instance of LEM is provable:

$$
\begin{array}{l}
2^{100} \text{ is the sum of two primes} \\
\qquad\qquad\qquad\quad \vee \\
\neg(2^{100} \text{ is the sum of two primes})
\end{array}
$$

- We do not currently have a proof of either disjunct, but we know that one of the disjuncts is **provable**
- As a result, the disjunction is also provable, and is thus **assertible**

An Infinite Generalisation

- The same goes for **every** instance of Goldbach's Conjecture: LEM holds for every single instance
- LEM only fails when we stop considering instances, and look at the fully general version of Goldbach's Conjecture
  - **Goldbach's Conjecture:** *Every* even number greater than 2 is the sum of two primes
- There are infinitely many even numbers, and as a result, we cannot go through each even number and check whether it is the sum of two primes

An Infinite Generalisation

- Of course, if we start checking even numbers, then we might find one which is not the sum of two primes
- That would be great, because it would amount to a proof of $\neg G$
- But it may be that every even number we check *is* the sum of two primes
- In that case, we would keep proving instance after instance of G, but that would not add up to a proof of G itself
  - No matter how far through the even numbers we go, there might always be a counterexample to Goldbach's Conjecture waiting around the corner

Finite versus Infinite Domains

- Let $\mathcal{F}$ be a **decidable predicate**
  - For each object in the domain, we can either prove that that object satisfies $\mathcal{F}$, or we can prove that it does not satisfy $\mathcal{F}$

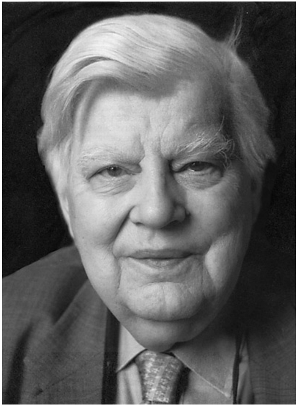
- When we are dealing with *finite* domains, this is enough to guarantee that it is decidable whether $\forall x \mathcal{F} x$
  - We will either be able to prove $\forall x \mathcal{F} x$, or we will be able to prove $\neg \forall x \mathcal{F} x$
  - Either way, we will be able to prove $\forall x \mathcal{F} x \vee \neg \forall x \mathcal{F} x$

- When we are dealing with *infinite domains*, this is not enough to guarantee that it is decidable whether $\forall x \mathcal{F} x$
  - We may not be able to prove either $\forall x \mathcal{F} x$ or $\neg \forall x \mathcal{F} x$
  - As a result, we may not be able to prove $\forall x \mathcal{F} x \vee \neg \forall x \mathcal{F} x$

The BHK Semantics

Dummett's Semantic Arguments

- Dummett is one of the most important advocates of IL there has ever been
- Dummett argued for IL by arguing that the fundamental semantic notion is WARRANTED ASSERTIBILITY, not TRUTH
- Dummett presented two related arguments
  - The Manifestation Argument
  - The Acquisition Argument
- We will focus on the Manifestation Argument



Michael Dummett

The Form of the Manifestation Argument

- The heart of Dummett's argument is a *reductio ad absurdum*
  - Dummett starts off by assuming that TRUTH is the fundamental semantic concept
  - He then attempts to derive an absurd result from this assumption
  - He ends the *reductio* by concluding that TRUTH is **not** the fundamental semantic concept
- After the *reductio*, Dummett simply proposes that ASSERTIBILITY would be a more fruitful candidate for the fundamental semantic concept

From Understanding to Knowing Truth-Conditions

- Assume that TRUTH is the fundamental semantic concept
- Presumably that means that to understand a sentence, you must know its truth-conditions
- For example, to understand Goldbach's Conjecture, G, is to know the conditions under which it would be true

Manifesting your Understanding

- **Dummett's Big Idea:** Whatever exactly your understanding of G consists in, it is essential that this understanding be fully *manifestable*
  - You must be able to demonstrate that you have the knowledge which would underlie an understanding of G
- This is crucial because meaning is fundamentally **public**
  - The meaning of our sentences is what we communicate to each other, and so it must be possible to make that meaning publicly available
- So if our understanding of G consists in knowing its truth-conditions, then we must have some way of manifesting that knowledge

Verification-Transcendent Truth-Conditions

- Suppose that G is undecidable: it is not possible to prove G, and it is not possible to prove $\neg G$
- In that case, the truth-conditions for G are **verification-transcendent**
  - It is beyond our means to verify whether G is true
- **Dummett's Question:** How would you ever *manifest* your knowledge of the *verification-transcendent* truth-conditions for G?

Stating Truth-Conditions

- You might think that you could manifest your knowledge of G's truth-conditions by *explicitly stating what they are*:
  - G is true iff: for all n, if n is a multiple of 2 then there are some numbers, j and k, such that each of these numbers is only divisible by 1 and itself, and $n = j + k$

- The trouble with this strategy is that you end up **using** a sentence which has verification-transcendent truth-conditions to state G's verification-transcendent truth-conditions

- This will not be much help to you if you were worried about how you could manifest an understanding of sentences with verification-transcendent truth-conditions!

Manifestation through Use

- Ultimately, your understanding of a sentence can only be manifested through the way that you **use** it
- So if to understand G is to know its truth-conditions, then you must be able to manifest this knowledge through the way that you use G
- But what use could manifest knowledge of **verification-transcendent** truth-conditions?

Replacing Truth with Proof

- Dummett does not think that there is any good answer to this question, and so concludes that TRUTH cannot be the fundamental semantic concept

  (Or at least, it can't be in mathematical discourse)

- Dummett recommends that we take PROOF as our fundamental semantic concept for mathematical discourses

- While the *truth-conditions* for G may be verification-transcendent, the *proof-conditions* are not
  - We know what it would take to prove G, and we can manifest that knowledge in various ways
  - For example, we can look over putative proofs of G, and judge whether they are successful

Tomorrow's Seminar

- For tomorrow's seminar, please read:
  - *An Intuitionistic Logic Primer*, §§5–6
  - Michael Dummett, 'The Philosophical Basis of Intuitionsitic Logic'
- The paper by Dummett is one of the places where he develops his Manifestation and Acquisition Arguments for IL
- Please bring *at least* one question to the seminar that you would like to discuss

Next Week's Lecture and Seminar

- Next week, we will look at Fitch's Paradox, which uses the resources of Modal Logic and Second-Order Logic to argue against Intuitionistic Logic!
- **Required Reading**
  - Timothy Williamson, 'Intuitionism Disproved?'
  - Dorothy Edgington, 'The Paradox of Knowability'
  - Michael Dummett, 'Victor's Error'
- All three of these articles are very short, and they are all available via the Reading List on the VLE
- Questions will shortly be posted on the VLE. Please bring written answers to these questions to the seminar