Intermediate Logic Spring
Lecture Seven

Natural Deduction for Intuitionistic Logic

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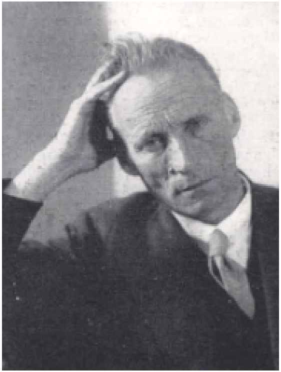
Natural Deduction for Intuitionisitic Logic

Restricting Classical Logic

- So far, we have looked at two non-classical logics
  - Modal Logic
  - Second-Order Logic
- Both of these were **extensions** of Classical Logic (CL)
  - They took CL, and then added some extra resources to it
- This week we are going to look at **Intuitionistic Logic** (IL)
- IL is a **restriction** of CL, not an extension of it!
  - IL takes CL, and *removes* some of its resources

The Origins of Intuitionism

- Intuitionism started life as a philosophy of mathematics, invented by L.E.J. Brouwer
- According to Brouwer, numbers are in some sense **constructed** by the mind
- In particular, we construct them within our faculty of *intuition*, hence the name **intuitionism**

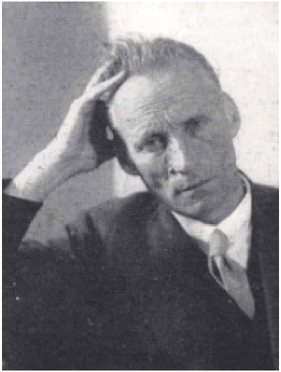


L.E.J. Brouwer

The Origins of Intuitionism

- This conception of mathematics led Brouwer (and his student Heyting) to revise Classical Logic
- In this module, we will set the philosophy of mathematics to one side, and focus on the logic

  (This logic is also sometimes known as **constructive** logic)



L.E.J. Brouwer

The Language of IL

The language of IL is exactly the same as the language of FOL

Rejecting a Basic Rule of FOL

- The difference between IL and FOL shows up in their natural deduction systems
- The system for IL includes all of the basic rules for FOL, **apart from TND**

i	$\quad\mid\ \mathcal{A}$	
j	$\quad\mid\ \mathcal{B}$	
k	$\quad\mid\ \neg\mathcal{A}$	
l	$\quad\mid\ \mathcal{B}$	
	$\mathcal{B}$	TND, i–j, k–l

Rejecting Derived Rules of FOL

- If we reject the basic rule of TND, then we have to reject a number of derived rules too
- Most obviously, we have to reject DNE:

m	$\neg\neg\mathcal{A}$	
	$\mathcal{A}$	DNE, m

Rejecting Derived Rules of FOL

- We also have to reject one of the De Morgan Rules

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
	$\neg\mathcal{A} \vee \neg\mathcal{B}$	DeM, m

- But we get to keep the other three De Morgan Rules!

Rejecting Derived Rules of FOL

- We also have to reject one of the rules for Converting Quantifiers

m	$\neg\forall x\mathcal{A}$	
	$\exists x\neg\mathcal{A}$	CQ, m

- But we get to keep the other three rules for Converting Quantifiers!

Natural Deduction for IL

- And that's it!
- All of the other rules for FOL listed in *forallχ*, basic and derived, carry over to IL
- As ever, we will use $\vdash$ to express *provability*, but we will add subscripts to indicate whether we are working with IL or classical FOL
  - $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash_I \mathcal{C}$ **iff** $\mathcal{C}$ can be proved from $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$, using only the rules of IL
  - $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash_C \mathcal{C}$ **iff** $\mathcal{C}$ can be proved from $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$, using any of the rules of classical FOL

Natural Deduction for Intuitionisitic Logic

Rejecting the Law of the Excluded Middle

- It is often said that intuitionists reject the **Law of the Excluded Middle** (LEM):

$$\mathcal{A} \vee \neg\mathcal{A}$$

- That is absolutely right, but it is important to be clear on what it really means

A Schematic Law

- LEM is a **schematic** law of CL
- This means that every *instance* of LEM is a theorem of CL
  - To build an instance of LEM, simply substitute the same sentence for both of the $\mathcal{A}$s in $\mathcal{A} \vee \neg\mathcal{A}$
- **Examples:**

$$P \vee \neg P$$

$$(P \vee Q) \vee \neg(P \vee Q)$$

$$\exists y \forall x(Fy \leftrightarrow x = y) \vee \neg\exists y \forall x(Fy \leftrightarrow x = y)$$

¬LEM

- You can reject LEM without accepting the negation of LEM as a new law
  - **LEM**: $\mathcal{A} \lor \neg\mathcal{A}$
  - ¬**LEM**: $\neg(\mathcal{A} \lor \neg\mathcal{A})$
- Clearly, you can deny that every instance of LEM is a theorem of logic without accepting that every instance of ¬LEM is a theorem!
- More surprisingly, intuitionists do not accept *any* instance of ¬LEM as a theorem
- In fact, you can prove that ¬LEM is a **contradiction** in IL

$\neg(\mathcal{A} \vee \neg\mathcal{A}) \vdash_I \bot$

1	$\neg(\mathcal{A} \vee \neg\mathcal{A})$	
2	$\quad \mathcal{A}$	
3	$\quad \mathcal{A} \vee \neg\mathcal{A}$	$\vee$I, 2
4	$\quad \bot$	$\bot$I, 3, 1
5	$\neg\mathcal{A}$	$\neg$I, 2–4
6	$\mathcal{A} \vee \neg\mathcal{A}$	$\vee$I, 5
7	$\bot$	$\bot$I, 6, 1

What it Means to Reject LEM

- When an intuitionist rejects LEM, all they are doing is denying that all of its instances are *logical theorems*
  - **In other words:** they are denying that it is always possible to proven an instance of LEM without the help of any premises
- That is quite right, in IL

  $\nvdash_I \mathcal{A} \vee \neg\mathcal{A}$

- The crucial point, then, is that there are *theorems* of classical FOL which are *not* theorems of IL
  - **Another example:** $((\mathcal{A} \to \mathcal{B}) \to \mathcal{A}) \to \mathcal{A}$ (aka Peirce's Law)

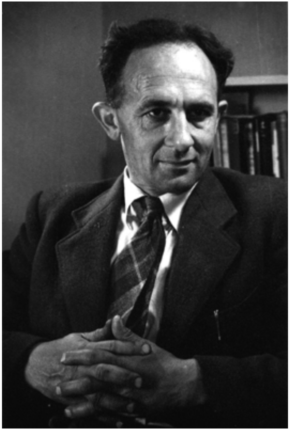
Why be an Intuitionist?

- Hopefully you now have a fair sense of how IL works
  - For more details, and for practice exercises too, look at the *Intuitionistic Logic Primer* on the VLE
- **Question:** Why be an intuitionist?
- This week we will look at an argument for intuitionism which focusses on the virtues of the natural deduction system for IL
- **But be warned:** this argument is a little bit indirect...

Natural Deduction for Intuitionisitic Logic

The Runabout Inference-Ticket

- We start with a short paper by Prior, called the 'Runabout Inference-Ticket'
- This paper wasn't really about intuitionism at all
- Prior was interested in an approach to logic known as **inferentialism**



Arthur Prior

The Rules for Conjunction

- Consider the natural deduction rules for conjunction

m	$\mathcal{A}$	
n	$\mathcal{B}$	
	$\mathcal{A} \wedge \mathcal{B}$	$\wedge$I, m, n

m	$\mathcal{A} \wedge \mathcal{B}$	
	$\mathcal{A}$	$\wedge$E, m

m	$\mathcal{A} \wedge \mathcal{B}$	
	$\mathcal{B}$	$\wedge$E, m

- **Question:** How do these rules relate to the *meaning* of '$\wedge$'?

Two Answers

- **Answer One**
  - These rules are *justified* by the meaning of '$\wedge$'
  - That meaning is fixed independently of the rules (perhaps by a truth-table), and the rules are required to conform to that meaning in the appropriate way

- **Answer Two: Inferentialism**
  - These rules *define* the meaning of '$\wedge$'
  - We do not need to justify these rules by showing that they conform to an independent meaning for '$\wedge$'
  - '$\wedge$' gets its meaning from these rules!

Prior versus Inferentialism

- Prior thought that inferentialism threatened to trivialise our whole deductive system
- **Prior's Assumption:** If inferentialism is true, then we can define a new logical connective with any combination of inferential rules
  - If the inferential rules *define* the connective, who is to stop us defining a connective with any rules we like?
- Prior then imagines defining a new connective, 'tonk', with the following rules

The Rules for 'Tonk'

m	$\mathcal{A}$	
	$\mathcal{A}$ tonk $\mathcal{B}$	tonk–I, m

m	$\mathcal{A}$ tonk $\mathcal{B}$	
	$\mathcal{B}$	tonk–E, m

- Essentially, 'tonk' has an one of the introduction rules for '$\vee$', and one of the elimination rules for '$\wedge$'
- **The Problem:** once you add 'tonk' to your system, you can prove any sentence from any sentence!

The Trivialisation Result

1	$\mathcal{A}$	
2	$\mathcal{A}$ tonk $\mathcal{B}$	tonk–I, 1
3	$\mathcal{B}$	tonk–E, 2

A Refutation of Inferentialism?

- Clearly, then, we cannot define a connective with the rules for 'tonk'
- Prior took this to be a refutation of inferentialism
  - If inferentialism were true, we would be able to define a new connective with any combination of rules
  - In that case, 'tonk' would be a perfectly good connective
  - But 'tonk' isn't a perfectly good connective
  - So inferentialism is false!

Natural Deduction for Intuitionisitic Logic

Defending Inferentialism

- Inferentialists have to reject **Prior's Assumption**
  - If inferentialism is true, then we can define a new logical connective with any combination of inferential rules

- **In other words:** even though a connective is defined by its inferential rules, we cannot use *any old* combination of rules to define a new connective

- Some combinations simply do not define a coherent meaning for a connective
  - The rules for 'tonk' do not manage to define a coherent meaning for 'tonk'

Introducing Harmony

- **Question:** Why do the rules for 'tonk' fail to define a coherent meaning?
- **One Answer:** Because the rules for 'tonk' are not in *harmony* with each other
- A connective's introduction rules and elimination rules should perfectly balance each other
- You shouldn't be able to get any more out of a connective by eliminating it than you have to put in to introduce it

  (You also shouldn't get any *less* out than you put in)
- Clearly, the rules for 'tonk' do let you get more out than you put in

No Precise Definition of ‘Harmony’

- Can we make this intuitive talk of ‘harmony’ more precise?
- In an ideal world, we would like to find a set of **necessary and sufficient** conditions for harmony
  - These would be conditions that are met by all and only the harmonious sets of inferential rules
- Unfortunately, no one has been able to come up with a set of necessary and sufficient conditions

A Necessary Condition for Harmony

- Happily, however, many philosophers and logicians *have* settled on a **necessary** condition for harmony
  - This is a condition which is satisfied by every harmonious set of rules

- **Guiding Idea:** If the introduction and elimination rules for a connective \$ are in harmony, then you shouldn't be able to prove anything new just by introducing \$ and then eliminating it

Local Peaks

- A **local peak** for \$ is a use of \$–I followed by a use of \$–E; the first line cited in the \$–E is the last line of the \$–I
- Here is an example of a local peak for '$\to$':

1	P (premise)	
2	P (assumption)	
3	$P \lor Q$	$\lor$I, 2
4	$P \to (P \lor Q)$	$\to$I, 2–3
5	$P \lor Q$	$\to$E, 4, 1

Levelling Local Peaks

- If $ is governed by harmonious introduction and elimination rules, then there must be a procedure for levelling any local peak for $

- A method for **levelling** local peaks for $ is a general method for re-writing proofs that include a local peak for $ in a way that eliminates that local peak

- So if $ is governed by harmonious rules, it must always be possible to eliminate any local peak for $ from a proof

Levelling Local Peaks for '$\rightarrow$'

i	$\mathcal{A}$		
j		$\mathcal{A}$	
$\ldots$		$\ldots$	
k		$\mathcal{B}$	
l	$\mathcal{A} \rightarrow \mathcal{B}$		$\rightarrow$I, j–k
m	$\mathcal{B}$		$\rightarrow$E, l, i

Levelling Local Peaks for '$\to$'

i	$\mathcal{A}$
$\dots$	$\dots$
j	$\mathcal{B}$

Levelling Local Peaks for '$\to$'

1	P	
2	P	
3	$P \vee Q$	$\vee$I, 2
4	$P \to (P \vee Q)$	$\to$I, 2–3
5	$P \vee Q$	$\to$E, 4, 1

Levelling Local Peaks for '$\to$'

1	P	
2	$P \vee Q$	$\vee$I, 1

Local Peaks for ‘tonk’

- This is what a local peak for ‘tonk’ looks like

j	$\mathcal{A}$	
k	$\mathcal{A}$ tonk $\mathcal{B}$	tonk–I, j
l	$\mathcal{B}$	tonk–E, k

- Since $\mathcal{A}$ and $\mathcal{B}$ can be *any* two sentences we like, there cannot be a general procedure for levelling local peaks for ‘tonk’
- So ‘tonk’ does not pass the necessary condition for harmony

Natural Deduction for Intuitionisitic Logic

Bringing it Back to Intuitionism

- **Question:** *What does any of this have to do with intuitionism!?*
- It turns out that the classical rules for negation do not pass our necessary condition for harmony!
- So if admissible rules are harmonious rules, the classical rules for negation must be abandoned!

Swapping TND for DNE

- The first thing we need to do is shift our focus from TND to DNE
  - DNE is a Negation Elimination rule, and harmony is all about balancing introduction and elimination rules

- *This is not a cheat!*
  - In *forall*χ, we took TND to be a basic classical rule, and then derived DNE
  - But we could just as well take DNE as the basic classical rule, and then derive TND

- So for present purposes, we will think of classical negation as being governed by the following three rules

Rules for Negation

$$
\begin{array}{r|l l}
m & \quad \mathcal{A} & \\
n & \quad \bot & \\
 & \neg\mathcal{A} & \neg\mathrm{I},\ m\text{–}n
\end{array}
$$

$$
\begin{array}{r|l l}
m & \mathcal{A} & \\
n & \neg\mathcal{A} & \\
 & \bot & \bot\mathrm{I},\ m,\ n
\end{array}
$$

$$
\begin{array}{r|l l}
m & \neg\neg\mathcal{A} & \\
 & \mathcal{A} & \mathrm{DNE},\ m
\end{array}
$$

Local Peaks for Classical Negation

- Since we have two Negation Elimination rules, there are two kinds of local peak for '¬'

- The kind which cause trouble are the ones which use DNE:

i	| $\neg\mathcal{A}$ (assumption)	
$\ldots$	| $\ldots$	
j	| $\bot$	
k	$\neg\neg\mathcal{A}$	¬I, i–j
l	$\mathcal{A}$	DNE, k

Local Peaks for Classical Negation

i	$\quad\mid \neg\mathcal{A}$	
$\dots$	$\quad\mid \dots$	
j	$\quad\mid \bot$	
k	$\neg\neg\mathcal{A}$	$\neg$I, i–j
l	$\mathcal{A}$	DNE, k

- There is no general procedure for levelling these kinds of local peak
- So the full classical rules for negation are not harmonious!

Intuitionistic Negation

- By contrast, there is a general procedure for levelling local peaks for '$\neg$', *when '$\neg$' is governed only by the intuitionistic rules!*
- In IL, '$\neg$' is governed only by $\neg$I and $\perp$I
- Since there is just one introduction rule and one elimination rule, all the local peaks look the same

Local Peaks for Intuitionistic Negation

i	$\mathcal{A}$		
j		$\mathcal{A}$	
$\ldots$		$\ldots$	
k		$\bot$	
l	$\neg\mathcal{A}$		$\neg$I, j–k
m	$\bot$		$\bot$I, i, l

Local Peaks for Intuitionistic Negation

i	$\mathcal{A}$
$\ldots$	$\ldots$
j	$\bot$

Is Intuitionistic Negation Harmonious?

- Does this prove that the intuitionistic rules for negation are harmonious?
- No — having a procedure for levelling local peaks is just a **necessary** condition for harmony, not a sufficient one
- However, intuitionistic negation certainly seems to be doing better than classical negation
  - The intuitionistic rules for negation pass this necessary condition
  - The classical rules for negation fail it!

Tomorrow's Seminar

- For tomorrow's seminar, please read:
  - *An Intuitionistic Logic Primer*, §§1–4
  - A.N. Prior, 'The Runabout Inference Ticket'
  - Nuel D. Belnap, 'Tonk, Plonk and Plink'
- Please note down any questions you have about these readings, *and bring them in writing to the seminar*

Next Week's Lecture and Seminar

- Next week, we will start looking at the semantics for Intuitionistic Logic
- **Required Reading**
  - *An Intuitionistic Logic Primer*, §§5–6
  - Michael Dummett, 'The Philosophical Basis of Intuitionsitic Logic'
- Both of these are available via the VLE
- Please note down any questions you have about these readings, *and bring them in writing to the seminar*