

Intermediate Logic Spring

Lecture Five

Second-Order Logic

Rob Trueman
rob.trueman@york.ac.uk

University of York

Second-Order Logic

Introduction

The Language of SOL

Introduction and Elimination Rules

Comprehension

Semantics

Identity in SOL

Two Arguments

Bertrand is a logician

Bertrand is a mathematician

∴ Someone is both a logician and a mathematician

Bertrand is a philosopher

Alfred is a philosopher

∴ Bertrand and Alfred have something in common

Two Arguments

Lb

Mb

$\therefore \exists x(Lx \wedge Mx)$

Bertrand is a philosopher

Alfred is a philosopher

\therefore Bertrand and Alfred have something in common

Two Arguments

 Lb Mb $\therefore \exists x(Lx \wedge Mx)$ Pb Pa $\therefore \exists X(Xb \wedge Xa)$

Introducing Second-Order Logic

- When we introduce variables that go where predicates go, we take the step from first-order logic (FOL) to *second-order logic* (SOL)
- The quantifiers in FOL let us quantify over **objects**
 - ‘ $\exists x(Lx \wedge Mx)$ ’ says that there is some *object* which is both L and M
- The quantifiers in SOL let us quantify over *properties*
 - ‘ $\exists X(Xb \wedge Xa)$ ’ says that there is some *property* which b and a both have

Second-Order Logic

Introduction

The Language of SOL

Introduction and Elimination Rules

Comprehension

Semantics

Identity in SOL

An Extension of FOL

- To get from FOL to SOL, all we need to do add some “second-order variables”
 - A **first-order variable** is a variable which can go where *names* go
 - A **second-order variable** is a variable which can go where *predicates* go
- We will use capital letters from the end of the alphabet, with or without numerical subscripts, as our second-order variables
 - S, T, U, V, W, X, Y, Z
 - $S_1, T_1, U_1, V_1, W_1, X_1, Y_1, Z_1$
 - $S_2, T_2, U_2, V_2, W_2, X_2, Y_2, Z_2$
 - ...

Monadic SOL

- As you already know, predicates come with a certain number of 'places'
 - A *one-place* predicate combines with one term at a time
 - A *two-place* predicate combines with two terms at a time
 - An *n-place* predicate combines with n terms at a time
- We *can* divide second-order variables up in exactly the same way...
- ...but to keep things simple, we will assume that *all* the second-order variables have just one place
 - Second-order variables can only combine with one term at a time

Formulas and Sentences of SOL

- You can build a new formula out of an old one by replacing one-place predicates with second-order variables

$$\neg Fa \Rightarrow \neg Xa$$

$$\forall x \forall y (Px \leftrightarrow Py) \Rightarrow \forall x \forall y (Yx \leftrightarrow Yy)$$

- You can build a new formula out of an old one by binding free second-order variables with quantifiers

$$\neg Xa \Rightarrow \forall X \neg Xa$$

$$\forall x \forall y (Yx \leftrightarrow Yy) \Rightarrow \exists Y \forall x \forall y (Yx \leftrightarrow Yy)$$

- A **sentence** of SOL is just a formula which contains no free variables (first-order or second-order)

Second-Order Logic

Introduction

The Language of SOL

Introduction and Elimination Rules

Comprehension

Semantics

Identity in SOL

An Extension of FOL

- The natural deduction system for SOL is an extension of the system for FOL
 - SOL includes **all** of the rules of FOL, basic and derived
- SOL simply adds some extra rules to govern the *second-order quantifiers*
 - **Second-order quantifiers** are quantifiers which bind second-order variables
- We start with their introduction and elimination rules, which look pretty much exactly the same as the rules for the first-order quantifiers

First-Order Universal Elimination

$$m \quad \left| \begin{array}{l} \forall \chi \mathcal{A}(\dots \chi \dots \chi \dots) \\ \mathcal{A}(\dots c \dots c \dots) \end{array} \right. \quad \forall_1 E, m$$

- $\mathcal{A}(\dots \chi \dots \chi \dots)$ is a formula containing **one or more** occurrences of some first-order variable χ
- c can be any name you like
- $\mathcal{A}(\dots c \dots c \dots)$ is the result of replacing **all** of the occurrences of χ in $\mathcal{A}(\dots \chi \dots \chi \dots)$ with c

Second-Order Universal Elimination

$$m \quad \left| \begin{array}{l} \forall X \mathcal{A}(\dots X \dots X \dots) \\ \mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots) \end{array} \right. \quad \forall_2 E, m$$

- $\mathcal{A}(\dots X \dots X \dots)$ is a formula containing **one or more** occurrences of some second-order variable X
- \mathcal{F} can be any one-place predicate you like
- $\mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots)$ is the result of replacing **all** of the occurrences of X in $\mathcal{A}(\dots X \dots X \dots)$ with \mathcal{F}

$$\forall Z(Za \rightarrow Zb), Ga \vdash_2 Gb$$

1		$\forall Z(Za \rightarrow Zb)$	
2		Ga	
		<hr/>	
3		$Ga \rightarrow Gb$	$\forall_2 E, 1$
4		Gb	$\rightarrow E, 3, 2$

First-Order Existential Introduction

$$m \quad \left| \begin{array}{l} \mathcal{A}(\dots c \dots c \dots) \\ \exists \chi \mathcal{A}(\dots \chi \dots c \dots) \end{array} \right. \quad \exists_1 I, m$$

- $\mathcal{A}(\dots c \dots c \dots)$ is a sentence containing **one or more** occurrences of the name c
- χ can be any first-order variable that does **not** occur in $\mathcal{A}(\dots c \dots c \dots)$
- $\mathcal{A}(\dots \chi \dots c \dots)$ is the result of replacing **one or more** of the occurrences of c in $\mathcal{A}(\dots c \dots c \dots)$ with χ

Second-Order Existential Introduction

$$m \quad \left| \begin{array}{l} \mathcal{A}(\dots\mathcal{F}\dots\mathcal{F}\dots) \\ \exists\mathcal{X}\mathcal{A}(\dots\mathcal{X}\dots\mathcal{F}\dots) \end{array} \right. \quad \exists_2I, m$$

- $\mathcal{A}(\dots\mathcal{F}\dots\mathcal{F}\dots)$ is a sentence containing **one or more** occurrences of a one-place predicate \mathcal{F}
- \mathcal{X} can be any second-order variable that does **not** occur in $\mathcal{A}(\dots\mathcal{F}\dots\mathcal{F}\dots)$
- $\mathcal{A}(\dots\mathcal{X}\dots\mathcal{F}\dots)$ is the result of replacing **one or more** of the occurrences of \mathcal{F} in $\mathcal{A}(\dots\mathcal{F}\dots\mathcal{F}\dots)$ with \mathcal{X}

$Pb, Pa \vdash_2 \exists X(Xa \wedge Xb)$

1		Pb	
2		Pa	
		<hr/>	
3		$Pb \wedge Pa$	$\wedge I, 1, 2$
4		$\exists X(Xa \wedge Xb)$	$\exists_2 I, 3$

First-Order Universal Introduction

$$\begin{array}{l|l}
 m & \mathcal{A}(\dots c \dots c \dots) \\
 & \forall \chi \mathcal{A}(\dots \chi \dots \chi \dots) \quad \forall_1 I, m
 \end{array}$$

- $\mathcal{A}(\dots c \dots c \dots)$ is a sentence containing one or more occurrences of the name c , and $\mathcal{A}(\dots \chi \dots \chi \dots)$ is the formula that you get when you replace **all** of those occurrences of c with the first-order variable χ
- c **must not** occur in any undischarged assumptions above line m (including the premises of the argument)
- c **must not** occur in $\forall \chi \mathcal{A}(\dots \chi \dots \chi \dots)$

Second-Order Universal Introduction

$$m \quad \left| \begin{array}{l} \mathcal{A}(\dots\mathcal{F}\dots\mathcal{F}\dots) \\ \forall\mathcal{X}\mathcal{A}(\dots\mathcal{X}\dots\mathcal{X}\dots) \end{array} \right. \quad \forall_2\text{I}, m$$

- $\mathcal{A}(\dots\mathcal{F}\dots\mathcal{F}\dots)$ is a sentence containing one or more occurrences of the one-place predicate \mathcal{F} , and $\mathcal{A}(\dots\mathcal{X}\dots\mathcal{X}\dots)$ is the formula that you get when you replace **all** of those occurrences of \mathcal{F} with the second-order variable \mathcal{X}
- \mathcal{F} **must not** occur in any undischarged assumptions above line m (including the premises of the argument)
- \mathcal{F} **must not** occur in $\forall\mathcal{X}\mathcal{A}(\dots\mathcal{X}\dots\mathcal{X}\dots)$

$$\vdash_2 \forall V \forall y (Vy \rightarrow Vy)$$

1	Fa	
2	Fa	R, 1
3	Fa → Fa	→I, 1–2
4	∀y(Fy → Fy)	∀ ₁ I, 3
5	∀V∀y(Vy → Vy)	∀ ₂ I, 4

First-Order Existential Elimination

$$\begin{array}{l|l}
 m & \exists x \mathcal{A}(\dots x \dots x \dots) \\
 n & \left| \begin{array}{l} \mathcal{A}(\dots c \dots c \dots) \\ \hline \mathcal{B} \end{array} \right. \\
 o & \left| \mathcal{B} \right. \\
 & \mathcal{B} \qquad \qquad \qquad \exists_1 E, m, n-o
 \end{array}$$

- c **must not** occur in any undischarged assumptions above line m (including the premises of the argument)
- c **must not** occur in $\exists x \mathcal{A}(\dots x \dots x \dots)$
- c **must not** appear in \mathcal{B}

Second-Order Existential Elimination

$$\begin{array}{l|l}
 m & \exists X \mathcal{A}(\dots X \dots X \dots) \\
 n & \left| \begin{array}{l} \mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots) \\ \hline \mathcal{B} \end{array} \right. \\
 o & \mathcal{B}
 \end{array}
 \quad \exists_2 E, m, n-o$$

- \mathcal{F} **must not** occur in any undischarged assumptions above line m (including the premises of the argument)
- \mathcal{F} **must not** occur in $\exists X \mathcal{A}(\dots X \dots X \dots)$
- \mathcal{F} **must not** appear in \mathcal{B}

$$\exists X(Xa \wedge Xb), \forall Y(Yb \rightarrow \neg Yc) \vdash_2 \exists Z(Za \wedge \neg Zc)$$

1	$\exists X(Xa \wedge Xb)$	
2	$\forall Y(Yb \rightarrow \neg Yc)$	
3	$Fa \wedge Fb$	
4	Fa	$\wedge E, 3$
5	Fb	$\wedge E, 3$
6	$Fb \rightarrow \neg Fc$	$\forall_2 E, 2$
7	$\neg Fc$	$\rightarrow E, 6, 5$
8	$Fa \wedge \neg Fc$	$\wedge I, 4, 7$
9	$\exists Z(Za \wedge \neg Zc)$	$\exists_2 I, 8$
10	$\exists Z(Za \wedge \neg Zc)$	$\exists_2 E, 1, 3-9$

Second-Order Logic

Introduction

The Language of SOL

Introduction and Elimination Rules

Comprehension

Semantics

Identity in SOL

Another Argument

Susanne is a pianist or an historian

Mary is a pianist or an historian

∴ Susanne and Mary have something in common

- This strikes me as a good argument
 - Even if Susanne isn't a historian and Mary isn't a pianist, they still have *something* in common: they are both pianists or historians!
- Unfortunately, the rules we have laid out so far will not allow us to provide a proof to vindicate this argument

Another Argument

$$Ps \vee Hs$$

$$Pm \vee Hm$$

$$\therefore \exists X(Xs \wedge Xm)$$

- This strikes me as a good argument
 - Even if Susanne isn't a historian and Mary isn't a pianist, they still have *something* in common: they are both pianists or historians!
- Unfortunately, the rules we have laid out so far will not allow us to provide a proof to vindicate this argument

Another Argument

$$Ps \vee Hs$$

$$Pm \vee Hm$$

$$\therefore \exists X(Xs \wedge Xm)$$

- This strikes me as a good argument
 - Even if Susanne isn't a historian and Mary isn't a pianist, they still have *something* in common: they are both pianists or historians!
- Unfortunately, the rules we have laid out so far will not allow us to provide a proof to vindicate this argument
- The trouble is that our rules only allow us to replace simple predicates, like, ' P ' and ' H ', with second-order variables, not complex formulas like ' $Px \vee Hx$ '

Comprehension

$$\left| \exists X \forall x (Xx \leftrightarrow \mathcal{A}(\dots x \dots x \dots)) \quad \text{Comp}$$

- X must not occur in $\mathcal{A}(\dots x \dots x \dots)$

$$Ps \vee Hs, Pm \vee Hm \vdash_2 \exists X(Xs \wedge Xm)$$

1	$Ps \vee Hs$	
2	$Pm \vee Hm$	
3	$\exists X \forall x (Xx \leftrightarrow (Px \vee Hx))$	Comp
4	$\forall x (Fx \leftrightarrow (Px \vee Hx))$	
5	$Fs \leftrightarrow (Ps \vee Hs)$	$\forall_1 E, 4$
6	Fs	$\leftrightarrow E, 5, 1$
7	$Fm \leftrightarrow (Pm \vee Hm)$	$\forall_1 E, 4$
8	Fm	$\leftrightarrow E, 7$
9	$Fs \wedge Fm$	$\wedge I, 6, 8$
10	$\exists X (Xs \wedge Xm)$	$\exists_2 I, 9$
11	$\exists X (Xs \wedge Xm)$	$\exists_2 E, 3, 4-10$

A Note on Comprehension

- You are allowed to plug in **any** formula for $\mathcal{A}(\dots \chi \dots \chi \dots)$
- *It can contain first-order quantifiers!*

$$\exists X \forall y (Xy \leftrightarrow \exists x Rxy)$$

- *It can even contain second-order quantifiers!!!*

$$\exists Y \forall y (Yy \leftrightarrow \forall x \exists X (Xx \vee Xy))$$

Natural Deduction for SOL

- **Our natural deduction system for SOL:**
 - All of the natural deduction rules for FOL
 - The Introduction and Elimination rules for the second-order quantifiers
 - Comprehension

(Logicians often add another rule to this system, called *Choice*, but we will leave that one out for now)

Second-Order Logic

Introduction

The Language of SOL

Introduction and Elimination Rules

Comprehension

Semantics

Identity in SOL

Set-Theory and Semantics

- The way I presented the semantics for FOL in *forall χ* was fairly informal
- Normally, philosophers use **set-theory** to formalise this semantics
- I explain this in some detail in the Primer, but I will go over a couple of the basics here
- This is important, because the **Standard Semantics** for SOL is set-theoretic
 - As you will see, not everyone agrees that the so-called “Standard Semantics” is the best semantics!

Set-Theorising our Interpretations

- In *forall* χ , I said that an interpretation specifies three things:
 - The referent of each name we are dealing with
 - The extension of each predicate we are dealing with
 - The domain of quantification
- We can think of the domain as a **set**, D
- We can also think of the extensions of our one-place predicates as **subsets** of D

$$A \subseteq B \leftrightarrow \forall x(x \in A \rightarrow x \in B)$$

The First-Order Quantifiers

- Let c be a new name added to the language
- $\forall x \mathcal{A}(\dots x \dots x \dots)$ is true in an interpretation iff $\mathcal{A}(\dots c \dots c \dots)$ is true in *every* interpretation that extends the original interpretation by assigning an object to c (without changing the interpretation in any other way)
- $\exists x \mathcal{A}(\dots x \dots x \dots)$ is true in an interpretation iff $\mathcal{A}(\dots c \dots c \dots)$ is true in *some* interpretation that extends the original interpretation by assigning an object to c (without changing the interpretation in any other way)

The Second-Order Quantifiers

- Let \mathcal{F} be a new one-place predicate added to the language
- $\forall X \mathcal{A}(\dots X \dots X \dots)$ is true in an interpretation iff $\mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots)$ is true in *every* interpretation that extends the original interpretation by assigning a subset of the domain to \mathcal{F} (without changing the interpretation in any other way)
- $\exists X \mathcal{A}(\dots X \dots X \dots)$ is true in an interpretation iff $\mathcal{A}(\dots \mathcal{F} \dots \mathcal{F} \dots)$ is true in *some* interpretation that extends the original interpretation by assigning a subset of the domain to \mathcal{F} (without changing the interpretation in any other way)

Logical Consequence

- $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models_2 C$ **iff** every interpretation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ true also makes C true
- Is \vdash_2 **sound** relative to \models_2 ?
 - If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \vdash_2 C$, then $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models_2 C$
- Is \vdash_2 **complete** relative to \models_2 ?
 - If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models_2 C$, then $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \vdash_2 C$

SOL is Inherently Incomplete!

- \vdash_2 is sound relative to \models_2 , **but it is not complete!**
- This is not just because I forgot to add some rules to our proof system for SOL
- It turns out that SOL is so powerful that no system of natural deduction for SOL can be both sound and complete!
 - Proving that is well beyond our means here, since it is a corollary of Gödel's Incompleteness Theorems
 - But if you are intrigued, then I recommend you take the Foundations of Mathematics module next term
 - You won't quite learn how to prove Gödel's theorems, but you will get a sense of what they mean, and why they are so important!

Second-Order Logic

Introduction

The Language of SOL

Introduction and Elimination Rules

Comprehension

Semantics

Identity in SOL

Defining Identity

- In FOL, identity is undefinable: you have to take it as a primitive, basic logical concept
- But in SOL, we can define identity!

$$\forall x \forall y (x = y \leftrightarrow_{df} \forall X (Xx \leftrightarrow Xy))$$

- **In English:** x is identical to y **iff** x and y have exactly the same properties

Two Leibnizian Principles

- This is an old definition of identity, which goes back to two principles proposed by Leibniz:
 - **Indiscernibility of Identicals:** $\forall x \forall y (x = y \rightarrow \forall X (Xx \leftrightarrow Xy))$
 - **Identity of Indiscernibles:** $\forall x \forall y (\forall X (Xx \leftrightarrow Xy) \rightarrow x = y)$
- The second-order definition of identity is what you get when you put these two principles together

The Identity of Indiscernibles

- You may have heard people say that while the Indiscernibility of Identical is unassailable, the Identity of Indiscernibles is a controversial thesis
- But in fact, it is quite easy to show that the Identity of Indiscernibles is true on every interpretation in the Standard Semantics
- The core of the argument runs like this:
 - Suppose ' a ' and ' b ' refer to distinct objects, 1 and 2
 - Now consider the sentence ' $Aa \leftrightarrow Ab$ '
 - This will come out false if we assign $\{1\}$ to ' A ' as its extension
 - So ' $\forall X(Xa \leftrightarrow Xb)$ ' will be false

Properties and Sets

- Does this show that all of the metaphysical debate about the Identity of Indiscernibles was a waste of time?
- *Not at all!*
- When I introduced you to SOL, I told you that second-order quantifiers quantify over **properties**
- But in the Standard Semantics, we swapped properties for **sets**
- This swap makes a lot of sense for formal purposes, because sets are well behaved mathematical objects
- But for metaphysical purposes, it is properties which really matter

Properties and Sets

- A metaphysician should think of the Standard Semantics as a mere model of what they are really interested in
 - The sets we assign to predicates merely *represent* the properties we really care about
- When we look at the Standard Semantics like that, we must ask: How well does our set-theoretic model represent reality?
- At this point, it becomes **very** interesting to ask whether it is possible for two distinct objects to share all of their properties!

Tomorrow's Seminar

- The reading for tomorrow's seminar is:
 - *A Second-Order Logic Primer*
- You can find this primer on the VLE

Next Week's Lecture and Seminar

- For next week's lecture and seminar, read:
 - W.V.O. Quine, *Philosophy of Logic*, pp. 64–68
 - George Boolos, 'To Be is to Be a Value of a Variable (or to Be Some Values of Some Variables)'
- A number of study questions will shortly be posted on the VLE; please bring **written** answers to all of these questions