Intermediate Logic Spring

Lecture One

Natural Deduction for Modal Logic

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Natural Deduction for Modal Logic

- Preliminaries
- What is Modal Logic?
- System K
- Possibility
- System T
- $S4$
- $S5$

Non-Classical Logics

- Last term, we studied **Classical Logic**
  - 'Classical Logic' is just what we call the standard logic which most philosophers and logicians are happy to use
- This term, we are going to look at three **non-classical logics**:
  - Modal Logic
  - Second-Order Logic
  - Intuitionistic Logic

Non-Classical Logics

- Modal Logic and Second-Order Logic are **extensions** of Classical Logic
  - They take everything that Classical Logic has to offer, and then add some more
- Intuitionistic Logic is a **restriction** of Classical Logic
  - It rejects certain classical rules of inference

Why Study Non-Classical Logics?

- Each one of these non-classical logics crops up a lot in philosophy, and so studying them now will help you a lot in your future studies
    - This is especially true of Modal Logic and Second-Order Logic, which philosophers help themselves to *all the time*

- But what is more, each one of these logics is interesting in its own right
    - They have interesting **formal** properties
    - They are surrounded by interesting **philosophical** issues

Teaching

- Contact Hours
  - 9 × 1 hour lectures (Thursday 12:00–13:00)
  - 9 × 1 hour seminar (Friday — check your timetable!)
  - Weekly Office Hours (Tuesday 10:30–11:30 & Thursday 15:30–16:30)
- Procedural Requirements
  - Attend lectures
  - Complete all required reading
  - Attend, and fully participate in, seminars

Logic Primers

- **There is no textbook for this module**
- However, you will be able to find short introductions to each of the non-classical logics on the VLE
  - These introductions are not full-fledged textbooks, but they will be enough to get you up to speed for this module
- If you are particularly interested in the formal properties of any of the logics we study, then you will be able to find references to proper textbooks in the introductions

The Reading List

- There is a full Reading List on the VLE site
- Readings marked **Essential** must be read in preparation for this module
- Readings marked **Recommended** would be good to read to get a fuller understanding of the material
- Readings marked **Background** are usually more advanced texts, and you only need to read them if you really want a deeper understanding

Seminars

- Some of the items on the Reading List are marked as **Seminar Reading**
- You **must** read these **before** the relevant seminar
- Not **every** seminar comes with reading; sometimes we will use seminars as an opportunity to do some exercises using the logics we are studying

Assessment

- **Summative Assessment**
  - 2,500 word essay
  - Due Monday Week 1, Summer Term
  - Worth 10 credits (50% of the Intermediate Logic module)
  - A list of questions will be posted on the VLE
- **Formative Assessment**
  - 500 word essay
  - E-mail to me (rob.trueman@york.ac.uk) by noon, Monday Week 6
  - Title: *What puzzles me the most is...*
  - You should lay out an issue that has been puzzling you, explain why it has been puzzling you, and then do your best to resolve that puzzle or difficulty

Assessment

- You will **not** be tested on your ability to prove things using any of the non-classical logics
- You will only be tested on your ability to engage with the philosophical issues surrounding the non-classical logics
- However, during this module we **will** look at how to prove things and construct counter-interpretations, for two reasons
  (1) Part of the aim of this module is to equip you to understand those philosophers who do use these non-classical logics
  (2) In order to understand the philosophical issues surrounding a non-classical logic, you need to have some understanding of how the logic actually works

Further Support

- Please feel free to e-mail me with any questions relating to this module (rob.trueman@york.ac.uk)

- And also make good use of the office hours
  - Tuesday 10:30–11:30 & Thursday 15:30–16:30
  - Feel free to come along in groups of up to four people

Natural Deduction for Modal Logic

What is Modal Logic?

- Modal Logic (ML) is the logic of **necessity** and **possibility**
- We use the symbol $\Box$ to express *necessity*
  - You can read $\Box\mathcal{A}$ as *It is necessarily the case that* $\mathcal{A}$
- We use the symbol $\Diamond$ to express *possibility*
  - You can read $\Diamond\mathcal{A}$ as *It is possibly the case that* $\mathcal{A}$

Varieties of Necessity

- There are lots of different kinds of necessity
  - It is **humanly impossible** for me to run at 100mph, but it is not **physically impossible** for me to move that fast
  - It is **physically impossible** for me to run faster than the speed of light, but it is not **logically impossible** for me to move that fast
- Which kind of necessity does ML deal with? *All of them!*
  - We start with a basic set of rules that govern $\Box$ and $\Diamond$
  - We then add more rules to fit whatever kind of necessity we are interested in

From TFL to ML

- The language of ML is an extension of TFL
  - We could have started with FOL, which would have given us Quantified Modal Logic (QML)
  - QML is much more powerful than ML, but it is also much more complicated

- The basic vocabulary of ML is exactly the same as the basic vocabulary of TFL, except it adds the symbols $\Box$ and $\Diamond$

- ML also has exactly the same rules for how to build sentences out of this vocabulary, but with a couple of extra rules for $\Box$ and $\Diamond$

Sentences of ML

(1) Every atom of ML is a sentence of ML

(2) If $\mathcal{A}$ is a sentence of ML, then $\neg\mathcal{A}$ is a sentence of ML

(3) If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \wedge \mathcal{B})$ is a sentence of ML

(4) If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \vee \mathcal{B})$ is a sentence of ML

(5) If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \rightarrow \mathcal{B})$ is a sentence of ML

(6) If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \leftrightarrow \mathcal{B})$ is a sentence of ML

(7) If $\mathcal{A}$ is a sentence of ML, then $\Box\mathcal{A}$ is a sentence of ML

(8) If $\mathcal{A}$ is a sentence of ML, then $\Diamond\mathcal{A}$ is a sentence of ML

(9) Nothing else is a sentence of ML

Natural Deduction for Modal Logic

System *K*

- We start with a particularly simple modal system called *K*, in honour of Saul Kripke
- As before, we will use ⊢ to express provability, but we will add a subscript '*K*' to indicate that we are using system *K*
  - You can prove $\mathcal{C}$ from $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n$ *in system K*
  - $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n \vdash_K \mathcal{C}$
- *K* includes all of the natural deduction rules from TFL, and then adds two more basic rules to govern $\Box$

Distribution

$$\begin{array}{l|ll} m & \Box(\mathcal{A} \to \mathcal{B}) & \\ & \Box\mathcal{A} \to \Box\mathcal{B} & \text{Dist, } m \end{array}$$

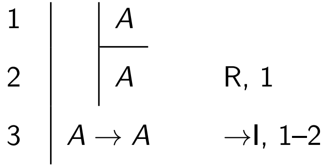
- This is known as the **Distribution Rule**, because it tells us that $\Box$ ‘distributes’ over $\to$

Necessitation

- **The basic idea:** if $\mathcal{A}$ is a theorem, then so is $\Box\mathcal{A}$
  - Remember, to say that $\mathcal{A}$ is a theorem is to say that $\mathcal{A}$ can be proved without relying on any undischarged assumptions
- This basic idea is easy enough to understand, and seems like quite a good rule
  - If you can **prove** $\mathcal{A}$ without relying on **any** assumptions, then surely it must be **necessarily** true!
- However, figuring out how to actually implement the Necessitation Rule in our proof system is a little tricky

Necessitation: An Easy Case

- Suppose we wanted to use Necessitation to prove $\Box(A \to A)$
- The first thing we need to do is prove that $A \to A$ is a theorem
- You already know how to do that using TFL: you simply present a proof of $A \to A$ which doesn't start wth any premises



1	A	
2	A	R, 1
3	$A \to A$	$\to$I, 1–2

Necessitation: An Easy Case

- Now that we have proven that $A \rightarrow A$ is a theorem, we should be able to apply Necessitation to infer $\Box(A \rightarrow A)$
- And in this case, there isn't really any problem:

$$
\begin{array}{r|l l}
1 & \quad \begin{array}{|l} A \\ \hline \end{array} & \\
2 & \quad \begin{array}{|l} A \end{array} & \text{R, 1} \\
3 & A \rightarrow A & \rightarrow\text{I, 1–2} \\
4 & \Box(A \rightarrow A) & \text{Nec, 3}
\end{array}
$$

Necessitation: A Difficult Case

- But now imagine that what we want to prove is $B \vdash_K B \wedge \Box(A \rightarrow A)$
- We might try something like this, but it would be no good:

1	B	
2	$\quad A$	
3	$\quad A$	R, 2
4	$A \rightarrow A$	$\rightarrow$I, 2–3
5	$\Box(A \rightarrow A)$	Nec, 4
6	$B \wedge \Box(A \rightarrow A)$	$\wedge$I, 1, 5

Necessitation: A Difficult Case

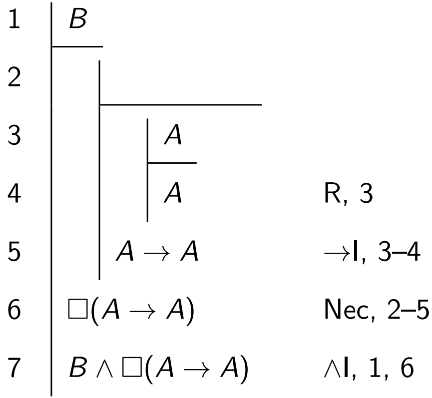
1	B	
2	A	
3	A	R, 2
4	$A \to A$	$\to$I, 2–3
5	$\Box(A \to A)$	Nec, 4
6	$B \wedge \Box(A \to A)$	$\wedge$I, 1, 5

- The trouble is our proof now starts with an undischarged assumption, B
- So all we really establish at line 4 is that $B \vdash_K A \to A$

Empty Assumptions

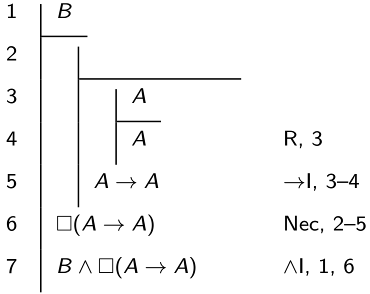
- To solve this problem, we need to find some way of showing that $A \to A$ is a theorem *in the middle of a longer proof*
- You are already familiar with the idea that you can trigger a new subproof whenever you like, just by making a new assumption
- We will now push that idea a little further, and say that you can also trigger a subproof by making an 'empty assumption'

Empty Assumptions



1	B	
2		
3	A	
4	A	R, 3
5	$A \to A$	$\to$I, 3–4
6	$\Box(A \to A)$	Nec, 2–5
7	$B \wedge \Box(A \to A)$	$\wedge$I, 1, 6

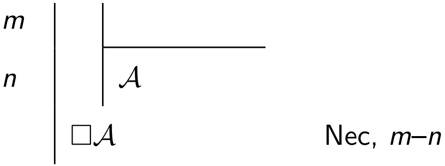
Empty Assumptions



1	B	
2		
3	A	
4	A	R, 3
5	$A \to A$	$\to$I, 3–4
6	$\Box(A \to A)$	Nec, 2–5
7	$B \land \Box(A \to A)$	$\land$I, 1, 6

- When we want to prove that something is a theorem, we start a subproof by making an ‘empty assumption’
- We then write out our proof of this theorem within the subproof

Necessitation: The Official Statement



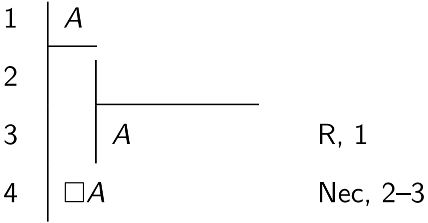
m | | (empty assumption line)

n | | $\mathcal{A}$

| $\Box\mathcal{A}$ Nec, m–n

- No line above line *m* may be cited by any rule within the subproof begun at line *m*.

A Bad Application of Necessitation



1	A	
2		
3	A	R, 1
4	$\Box A$	Nec, 2–3

- This is not a legitimate application of Necessitation, because at line 3 we appealed to line 1, which comes before the empty assumption at line 2

Some Results

- In system K, you can prove all of the following:
  1. $\Box(A \wedge B) \vdash_K \Box A \wedge \Box B$
  2. $\Box A \wedge \Box B \vdash_K \Box(A \wedge B)$
  3. $\Box A \vee \Box B \vdash_K \Box(A \vee B)$
  4. $\Box(A \leftrightarrow B) \vdash_K \Box A \leftrightarrow \Box B$
- We will go through some of these as exercises in the seminars, but let's look at how to prove 1 now

$\Box(A \wedge B) \vdash_K \Box A \wedge \Box B$

1	$\Box(A \wedge B)$	
2	$\mid$	
3	$\mid\ \mid\ A \wedge B$	
4	$\mid\ \mid\ A$	$\wedge$E, 3
5	$\mid\ (A \wedge B) \to A$	$\to$I, 3–4
6	$\Box((A \wedge B) \to A)$	Nec, 2–5
7	$\Box(A \wedge B) \to \Box A$	Dist, 6
8	$\Box A$	$\to$E, 7, 1

$\Box(A \wedge B) \vdash_K \Box A \wedge \Box B$

1	$\Box(A \wedge B)$	
2		
3	$A \wedge B$	
4	A	$\wedge$E, 3
5	$(A \wedge B) \to A$	$\to$I, 3–4
6	$\Box((A \wedge B) \to A)$	Nec, 2–5
7	$\Box(A \wedge B) \to \Box A$	Dist, 6
8	$\Box A$	$\to$E, 7, 1
9		
10	$A \wedge B$	
11	B	$\wedge$E, 10
12	$(A \wedge B) \to B$	$\to$I, 10–11
13	$\Box((A \wedge B) \to B)$	Nec, 9–12
14	$\Box(A \wedge B) \to \Box B$	Dist, 13
15	$\Box B$	$\to$E, 14, 1
16	$\Box A \wedge \Box B$	$\wedge$I, 8, 15

Natural Deduction for Modal Logic

What about Possibility?

- We have now gone over **all** of the basic rules of K
  - $K = \text{TFL} + \text{Dist} + \text{Nec}$
- But you might have noticed that these rules only deal with necessity ($\Box$)
- What happened to *possibility* ($\Diamond$)?

Defining Possibility

- It turns out that we can define possibility in terms of necessity:
  - $\Diamond\mathcal{A} =_{df} \neg\Box\neg\mathcal{A}$
- As a result, we do not really need a special symbol for possibility: we can get by just using $\Box$ and $\neg$
- Still, the system will be much easier to use if we do have a possibility symbol, and so we will add the following definitional rules

Defining Possibility

m	$\neg\Box\neg\mathcal{A}$	
	$\Diamond\mathcal{A}$	$\Diamond$Def, m

m	$\Diamond\mathcal{A}$	
	$\neg\Box\neg\mathcal{A}$	$\Diamond$Def, m

- Importantly, you should not think of these rules as any real addition to K
- They just record the way that $\Diamond$ is defined in terms of $\Box$

Modal Conversion

$$\begin{array}{r|ll} m & \neg\Box\mathcal{A} & \\ & \Diamond\neg\mathcal{A} & \text{MC}, m \end{array}$$

$$\begin{array}{r|ll} m & \Diamond\neg\mathcal{A} & \\ & \neg\Box\mathcal{A} & \text{MC}, m \end{array}$$

$$\begin{array}{r|ll} m & \Box\neg\mathcal{A} & \\ & \neg\Diamond\mathcal{A} & \text{MC}, m \end{array}$$

$$\begin{array}{r|ll} m & \neg\Diamond\mathcal{A} & \\ & \Box\neg\mathcal{A} & \text{MC}, m \end{array}$$

- All of these Modal Conversion rules can be derived from the basic rules of K, plus $\Diamond$Def

$\neg\Box A \vdash_K \Diamond\neg A$

1	$\neg\Box A$	
2		
3	$\neg\neg A$	
4	A	DNE, 3
5	$\neg\neg A \to A$	$\to$I, 3–4
6	$\Box(\neg\neg A \to A)$	Nec, 3–5
7	$\Box\neg\neg A \to \Box A$	Dist, 6
8	$\neg\Box\neg\neg A$	MT, 7, 1
9	$\Diamond\neg A$	$\Diamond$Def, 8

Natural Deduction for Modal Logic

The Limits of K

- K is a very simple system
- It is so simple, that it will not even let you infer $\mathcal{A}$ from $\Box\mathcal{A}$
  - In English: K will not let us infer that $\mathcal{A}$ is true from the assumption that $\mathcal{A}$ is **necessarily** true!
- Nor will it let us infer $\Diamond\mathcal{A}$ from $\mathcal{A}$
  - In English: K will not let us infer that $\mathcal{A}$ is **possibly** true from the assumption that $\mathcal{A}$ is **actually** true
- This leads us to a new system of ML, T, which we get by adding one new rule to K

The T Rule

$$\begin{array}{r|ll} m & \Box\mathcal{A} & \\ & \mathcal{A} & T,\ m \end{array}$$

From Actually-True to Possibly-True

- $T = K +$ the T Rule
- Clearly, T allows us to infer $\mathcal{A}$ from $\Box\mathcal{A}$
- But it turns out that it also allows us to infer $\Diamond\mathcal{A}$ from $\mathcal{A}$
  - $\mathcal{A} \vdash_T \Diamond\mathcal{A}$
- However, we will save the proof of that for the seminar!

Natural Deduction for Modal Logic

Preliminaries

What is Modal Logic?

System K

Possibility

System T

$S4$

$S5$

Adding Boxes

- System T allows you to strip away necessity boxes:
  - From $\Box\mathcal{A}$, you can infer $\mathcal{A}$
- But what if you wanted to *add extra* boxes?
  - Can you go from $\Box\mathcal{A}$ to $\Box\Box\mathcal{A}$?
- That would be no problem, **if you had proven $\Box\mathcal{A}$ by applying Necessitation**

$\vdash_T \Box(A \to A)$

1		
2	A	
3	A	R, 2
4	$A \to A$	→I, 2–3
5	$\Box(A \to A)$	Nec, 1–4

$\vdash_T \Box\Box(A \to A)$

1		
2		
3	A	
4	A	R, 3
5	$A \to A$	→I, 3–4
6	$\Box(A \to A)$	Nec, 2–5
7	$\Box\Box(A \to A)$	Nec, 1–6

But You Can't Always Add an Extra $\Box$ in T

- However, we do not always get $\Box\mathcal{A}$ by applying Necessitation
- It might be, for example, that $\Box\mathcal{A}$ is just an **assumption** that we made
- Are we always free to infer $\Box\Box\mathcal{A}$ from $\Box\mathcal{A}$?
- Not in T we're not, and that seems like a shortcoming of the system
  - It seems intuitive that if $\mathcal{A}$ is necessarily true, then it couldn't have *failed* to be necessarily true
- This leads us to another new system, $S4$, which we get by adding a new rule to T

The *S*4 Rule

$$\begin{array}{r|ll} m & \Box\mathcal{A} & \\ & \Box\Box\mathcal{A} & \qquad S4,\ m \end{array}$$

Deleting Diamonds

- $S4 = T +$ the $S4$ Rule
- As well as allowing us to *add* extra *boxes*, the $S4$ rule also lets us *delete* extra *diamonds*:
  - $\Diamond\Diamond\mathcal{A} \vdash_{S4} \Diamond\mathcal{A}$
- However, we will save the proof of that for the seminar!

Natural Deduction for Modal Logic

- Preliminaries
- What is Modal Logic?
- System K
- Possibility
- System T
- $S4$
- $S5$

Adding Boxes to Diamonds

- In *S*4, we can always add a box in front of another box
- But *S*4 does not automatically allow us to add a box in front of a *diamond*
  - *S*4 does not generally permit the inference from $\Diamond\mathcal{A}$ to $\Box\Diamond\mathcal{A}$
- But again, that might strike you as a shortcoming of *S*4
  - It seems intuitive that if $\mathcal{A}$ is possibly true, then it couldn't have *failed* to be possibly true
- This leads us to one last system, *S*5, which we get by adding a different rule to *T*

The $S5$ Rule

$$\begin{array}{l|ll}
m & \Diamond\mathcal{A} & \\
 & \Box\Diamond\mathcal{A} & S5,\ m
\end{array}$$

You Only Ever Need One Modal Operator

- $S5 = T +$ the $S5$ Rule
- As well as allowing us to *add* boxes in front of diamonds, the $S5$ rule also lets us *delete* diamonds in front of boxes:
  - $\Diamond\Box\mathcal{A} \vdash_{S5} \Box\mathcal{A}$
- And in fact, it also turns out that we can derive the $S4$ rule in $S5$:
  - $\Box\mathcal{A} \vdash_{S5} \Box\Box\mathcal{A}$
  - $\Diamond\Diamond\mathcal{A} \vdash_{S5} \Diamond\mathcal{A}$
- More generally, if you have a long string of boxes and diamonds, in any combination whatsoever, you can delete all but the last of them
  - For example: $\Diamond\Box\Diamond\Diamond\Box\Box\Diamond\Box A \vdash_{S5} \Box A$.

Tomorrow's Seminar

- The reading for tomorrow's seminar is:
  - *A Modal Logic Primer*, §§1–3
- Attempt all of the exercises in these sections, but try to resist the urge to look at the answers — we will be going through them in the seminars!

Next Week's Lecture and Seminar

- For next week's lecture and seminar, read:
  - *A Modal Logic Primer*, §4
- Attempt all of the exercises in this section, but try to resist the urge to look at the answers — we will be going through them in the seminars!