Quantification 4
Substitutional quantification and second-order quantification

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25/11/11

Last time

- We saw that the simple explanation of the quantifiers we gave in the second week was unable to deal with uncountable domains
- We then used the β-variant method to give a formal semantics for the quantifiers
- Lastly, we looked at Tarski's semantics for the quantifiers

Today's lecture

Unrestricted quantification

Substitutional quantification

Second-order quantification

Philosophical questions

- Believe it or not, we can now ask some genuinely interesting questions about quantification
    1. Can we quantify over *absolutely* everything?
    2. Do we *have* to quantify over a domain?
    3. Can bound variables *only* occur in positions that can be occupied by names?
- In this lecture we will focus on 2 and 3, but I will now briefly discuss 1.

A transcendental argument?

- It may seem as though we have a transcendental argument that we *must* be able to quantify over absolutely everything
- If we try to deny that we can quantify over absolutely everything, we say ‘We cannot quantify over absolutely everything’
- But ‘everything’ in ‘We cannot quantify over absolutely everything’ must quantify over absolutely everything
- Otherwise, ‘We cannot quantify over absolutely everything’ would not say that we quantify over absolutely everything
- So, when we try to deny that we can quantify over absolutely everything, we actually do quantify over absolutely everything, and hence refute ourselves

The limits of set theory

- It looks, then, like we must be able to quantify over absolutely everything
- But on the normal semantics for the quantifiers, we cannot
- We said that quantifiers quantify over *domains*, and domains are sets
- Remember in 1A that the standard response to Russell's paradox involves denying the existence of a universal set (i.e. of a set that has every object as a member)
- So, no domain can contain every object, and hence we cannot quantify over absolutely everything

Naturalising ontology

- Ontology is the attempt to answer the question ‘What is there?’
- A very difficult question: what is the method for giving a (non-trivial) answer to that question?
- We might have thought that there is some special philosophical method. The commitment to such a method is sometimes called ‘first-philosophy’ because it puts philosophy first
- Quine strongly opposed first-philosophy. He wanted to “naturalise ontology”
- He suggested the following method for answering the ontological question: we are ontologically committed to whatever objects are needed to make our best (scientific) theories true

  (Or in other words, we have to believe in the objects needed to make our best (scientific) theories true)

To be is to be the value of a variable

- Quine argued that we express the ontological commitments of a theory is by using the existential quantifier
- This position is summarised in Quine's famous slogan: To be is to be the value of a variable
- It is fair to say that this conception of ontological commitment has become orthodoxy
- It makes a lot of sense on the classical semantics for the quantifiers (i.e. the semantics we looked at in lecture 3). Recall that we quantify over a *set of objects*; it would be strange to then deny that we are committed to those objects existence!
- (We might, however, doubt Quine's belief that it is particularly scientific theories that we should look to)

Abandoning Quine

- But despite the orthodox status of Quine's conception of ontological commitment, there are dissenters
- Advocates of *substitutional quantification* argue that we need (in at least some cases) to the classical assumption that when we quantify we quantify over a domain of objects. We will call quantifying over a domain of objects *objectual quantification*.
- The crucial idea is that sentences involving proper names like '$F(a)$' can be true even if 'a' doesn't refer to anything
- So, 'Santa is a kindly old man with a big belly' can be true

Substitutional quantification

- Given this assumption, universal quantification can be understood as follows: '$\forall x \psi(x)$' is true iff '$\psi(a)$' is true for every name 'a'
- Likewise, '$\exists x \psi(x)$' is true iff '$\psi(a)$' is true for some name 'a'
- Note that the substitutional reading differs from the Fregean account considered in lecture 2. The Fregean account presupposes that all names have a reference. But the substitutional quantification advocate does not presuppose that names have to refer: 'Santa' is a perfectly good, though referentless name
- Clearly, this reading cuts all ties — so crucial on the Quinean story — between quantification and existence

Motivations

- We can quantify without being ontologically committed to thing we are quantifying over, which may be tempting in, e.g. fictional or mathematical contexts (e.g. '$\exists x$ x delivers presents to every child in one night')
- It also seems to allow us to quantify into contexts which would be (at least seemingly) illegitimate on the objectual semantics. For example, when we read the quantifier substitutionally, we can state the following principle about reference
  - $\forall x$(If 'x' refers to anything, then it refers to x)

  We cannot do that on the classical semantics as the first 'p' is mentioned, not used. Equally, it seems as though we can use substitutional quantification to quantify into intensional contexts, e.g.
  - $\exists x$(Lois believes that x has super powers)

Problems

- We still have the problem of not having enough names to simulate quantification over an uncountable domain
  - But you might wonder how worried an advocate of substitutional quantification would be, who doesn't really think in terms of domains at all
- Some clearly true quantified sentences seem to come out false when we read the quantifiers substitutionally: 'There is an object with no name in our language'
  - The advocate of substitutional quantification might reply that he is interested in extensions of our language
- The straightforward explanation of substitutional quantification involves objectual quantification (over names) in the metalanguage
  - If this really is a problem (it's not clear that it is), perhaps the advocate of substitutional quantification can give an *inferentialist* account of the quantifier

Introducing second-order quantification

- On the classical semantics, we can only quantify into name positions ("nominal positions")
- So, from 'Socrates is wise' we can infer '$\exists x\ x$ is wise'
- But don't we also want to be able to quantify into predicate positions?
- For example, we might think that from 'Socrates is wise' we can infer '$\exists X\ X$(Socrates)' (or 'Socrates is something' or 'Socrates has some property')
- Quantifying into predicate places is known as *second-order quantification*. (Quantifying into nominal places is known as *first-order quantification*)

The benefits of second-order quantification

- Second-order quantification allows us to handle intuitively valid inferences which cannot be captured in a first-order logic
- For example
  - Socrates is wise and Plato is wise
  - So, there is something which Socrates and Plato have in common

  is treated as
  - F(Socrates) and F(Plato)
  - So, $\exists X(X(\text{Socrates}) \text{ and } X(\text{Plato}))$
- Also, there are sentences which are inexpressible in first-order languages, e.g. 'Napoleon has all the properties of a great general'

A sketch of the "full" semantics for second-order logic

- The semantics for the first-order fragment of full second-order logic is just as it was last week
- We then add the following two clauses

  Let $\mathcal{I}$ be any interpretation of $\mathcal{L}$, 'ϕ' any sentence of $\mathcal{L}$, 'X' any *k*-place second-order variable of $\mathcal{L}$ and 'β' the first *k*-place predicate not to occur in 'ϕ'

  - If 'ϕ' = '$\forall X \psi$' then 'ϕ' is true on $\mathcal{I}$ iff '$\psi[X/\beta]$' is true on every β-variant of $\mathcal{I}$
  - If 'ϕ' = '$\exists X \psi$' then 'ϕ' is true on $\mathcal{I}$ iff '$\psi[X/\beta]$' is true on some β-variant of $\mathcal{I}$
- Crucially, on the full semantics for second-order logic, an interpretation can assign a *k*-place predicate *any* set of *k*-tuples of objects from the domain. For example, an interpretation can assign any subset in the domain to a one-place predicate

Is second-order “logic” logic?

- When Frege invented modern quantified logic, he used a second-order language. In fact, he used a “higher-order” language, with no (finite) upper limit on the level of quantification
- Equally, Alfred Whitehead and Bertrand Russell used a higher-order logic in their *Principia Mathematica*
- But despite its pedigree, second-order logic has fallen into disrepute. First-order logic is now considered uncontroversially logic, but second-order logic’s status is controversial

Quine's rejection of second-order logic

- Quine was one of the major characters in the demotion of second-order logic
- Quine argued that if variables can occur in predicate positions then predicates must refer to something, that over which the second-order variables range
- If those things are properties or attributes then we have a problem because we cannot postulate the existence of entities whose identity conditions are as nebulous as those of properties. (This is Quine's other famous slogan 'No entity without identity' at work)

Set theory in sheep's clothing

- If on the other hand we think that predicates refer to sets (as we do in the full semantics for second-order logic), then Quine complained that second-order logic is "set theory in sheep's clothing"
- Quine's point was that we cannot pretend that second-order logic is logic. A sentence like '$\exists X(Xa)$' is just another way of writing '$\exists x(a \in x)$'
- Logic is meant to be topic neutral, but second-oder logic has a subject matter: sets
- Quine claimed that we have crossed a boundary between logic and a mathematical theory

Soundness and completeness

- Some people have also appealed to the incompleteness of second-order logic as grounds for rejecting it as logic
- There are two ways to characterise entailment: syntactically and semantically
- A set of sentences Γ syntactically entail a sentence P ($\Gamma \vdash P$) iff the rules for manipulating the symbols of our logic allow us to get to P from Γ. An example is the tree method
- A set of sentences Γ semantically entail a sentence P ($\Gamma \models P$) iff there is no interpretation which makes all of the sentences of Γ true and P false
- Given that we have these two notions of entailment, we might be interested in how they relate to each other
- A logic is *sound* iff (if $\Gamma \vdash P$ then $\Gamma \models P$)
- A logic is *complete* iff (if $\Gamma \models P$ then $\Gamma \vdash P$)

The incompleteness of second-order logic

- Obviously we in general want a logic to be sound, and so design them with an eye to that feature. (Although it is tricky proving that a logic is sound!)
- But it would also be good if logics were complete
- Propositional logics (e.g. PL) are complete
- First-order logics (e.g. QL) are complete
- But full second-order logic is incomplete
- Some people have suggested that it is this difference which makes full second-order “logic” not a logic

Defending second-order logic

- First, it is not obviously right that on the full semantics for second-order logic, '$\exists X(Xa)$' is just another way of writing '$\exists x(a \in x)$'; it looks as though Quine is ignoring the object-language/meta-langauge distinction
- Second, it is not clear that Quine that appealing to sets prevents second-order logic being topic neutral. We normally think that we can take sets of any objects we like, and so set theory applies to every topic
- Third, second-order logic does not use all of the power of set theory

Still defending second-order logic

- What about the fact that second-order logic is incomplete?
- Well, it is not clear that that fact tells us that second-order logic is not logic
- Recall that PL is decidable and QL is not. That is a big difference, but we do not usually think that that stops QL being a logic. Why is completeness more significant than decidability?
- And anyway, it is only *full* second-order logic that is incomplete. There are weaker versions of second-order logic (Henkin semantics) that are complete
- The difference between full semantics and Henkin semantics is (roughly), that in the former but not the latter we have a guarantee that we quantify over *every* set of k-tuples of objects from the domain