Quantification 3
Formal semantics for the quantifiers

Rob Trueman
rt295@cam.ac.uk

Fitzwilliam College, Cambridge

18/11/11

Last time

- We introduced Frege's analysis of sentences into terms and predicates
- This analysis of sentences to describe the constructional history of sentences
- We then showed how thinking in terms of constructional histories allows us to give a uniform treatment of quantifier which applies whether they are within the scope of other quantifiers or not
- Lastly, we saw how moving to the quantifier-variable notation solves problems of scope

Today's lecture

Why do we need a formal semantics?

β-variants

Tarski

The simple explanation of the quantifiers

- We might well wonder what there is left to do
- Last week we said that very roughly,
    - $\forall x F(x)$

  is true iff all of
    - $F(\text{Joe})$
    - $F(\text{Rachel})$
    - $F(\text{Brian})$
    - ...

  are true
- And we said that very roughly,
    - $\exists x F(x)$

  is true iff at least one of
    - $F(\text{Joe})$
    - $F(\text{Rachel})$
    - $F(\text{Brian})$
    - ...

  is true
- Very rough indeed!

What's wrong with this picture?

- This explanation presupposes that there is a name for each object in the domain
- Couldn't we just add names for each object, at least for the purposes of idealisation?
- No. There are some domains so big that there just aren't enough names to go around
- Languages are *countable*, but we can have *uncountably* many objects in a domain

Another explanation

- Here is another simple, plausible way of dealing with quantification
- Take the sentence
  - $\forall x(x$ sang)

  and suppose that our domain consists of the objects Joe, Rachel and Brian
- 'Joe sang and Rachel sang and Brian sang' follows from '$\forall x(x$ sang)'
- And conversely, if we know that the domain is exhausted by Joe, Rachel and Brian, then '$\forall x(x$ sang)' follows from 'Joe sang and Rachel sang and Brian sang'
- In this case, therefore, '$\forall x(x$ sang)' is logically equivalent to 'Joe sang and Rachel sang and Brian sang'
- Similarly, the existential quantifier '$\exists x(x$ sang)' may be understood as the disjunction 'Joe sang or Rachel sang or Brian sang'

What's wrong with this (other) picture?

- Clearly this generalises to any finite domain. Therefore, quantifiers are superfluous in the context of determinate finite domains
- But we can't extend this idea to infinite domains. That is, we can't in general explain '$\forall x F(x)$' as shorthand for '$F(a_1) \wedge F(a_2) \wedge F(a_3) \ldots$'
  - This account requires that there be a conjunct for each object, and so as before requires that there be a name for each object in the domain
  - But even worse, this approach requires infinitely long sentences. It might reasonably be objected that so great an idealisation will not shed any light on natural languages
- So how are we to give an acceptable semantics for the quantifiers?

One way around the problem

- Take the sentence
  - $\forall x(x \text{ sang})$
- Rather than adding names for all of the objects in our domain into our language at once, let's just add one name, 'b', and replace the 'x' in 'x sang' with that name, giving us
  - b sang
- We can read 'b' as a name for any object in the domain we like. It can be a name for Joe, Rachel, Brian, or whatever else happens to be in our domain
- We can, then, say that '$\forall x(x \text{ sang})$' is true iff '$b$ sang' is true on all the different ways of reading the name 'b'
- Similarly, '$\exists x(x \text{ sang})$' is true iff '$b$ sang' is true on at least one of the ways of reading 'b'

There is nothing wrong with this picture

- This suggestion is very similar to the one we first mentioned last lecture
- Crucially, though, it does not require us to add a name for every object in our language. Instead, we just pick one name and then look at all the different ways of reading that name
- But we cannot leave matters here. We need to make it much more precise

The vocabulary of $\mathcal{L}$

- To make things simpler and more precise, we will move to a formal language $\mathcal{L}$ (Mates, *Elementary Logic*, ch.3)
- $\mathcal{L}$ has a very simple vocabulary:
  - Variables — 'x_1' 'x_2' 'x_3'...
  - Logical constants — '$\neg$' '$\vee$' '(' ')' '$\wedge$' '$\supset$' '$\forall$' '$\exists$'
  - Sentence letters — 'P_1' 'P_2' 'P_3'...
  - Predicates — capital italic 'F's with numerical subscripts and numerical superscripts for positive integers; e.g. 'F_1^1' 'F_2^2' 'F_{9384}^{101}'
  - Names (or individuals constants) — 'a_1' 'a_2' 'a_3'...

Formulae of $\mathcal{L}$

- A *predicate of degree n* (or an *n*-ary predicate) is a predicate having as a superscript a numeral for the positive integer *n*
- An *individual symbol* is a variable or name
- An *atomic formula* is an expression consisting either of a sentence letter or of an *n*-ary predicate followed by a string of *n* individual symbols
- We now move to the broader notion of a *formula*
    - An atomic formula is a formula
    - If 'ϕ' is a formula, then '$\neg\phi$' is a formula
    - If 'ϕ' and 'ψ' are formulae, then '$(\phi \wedge \psi)$', '$(\phi \vee \psi)$' and '$(\phi \supset \psi)$' are formulae
    - If 'ϕ' is a formula and 'α' is a variable, then '$\forall\alpha\phi$' and '$\exists\alpha\phi$' are formulae
    - Nothing else is a formula

Sentences of $\mathcal{L}$

- An occurrence of a variable 'α' in 'ϕ' is *bound* iff it is within an occurrence in 'ϕ' of a formula of the form '$\forall\alpha\psi$' or of the form '$\exists\alpha\psi$'; otherwise it is a *free* occurrence of 'α'
    - 'x_1' occurs bound in '$\forall x_1 F_1^1 x_1$' and '$(\forall x_1 F_1^1 x_1 \wedge P)$'
    - 'x_1' occurs free in '$F_1^1 x_1$' and '$\forall x_2 F_1^2(x_2, x_1)$'
    - There is one free occurrence and one bound occurrence of 'x_1' in '$(\forall x_1 F_1^1 x_1 \wedge F_2^1 x_1)$'
- A *sentence* is a formula in which no variable occurs free. An *atomic sentence* is an atomic formula in which no variable occurs free
- Sometimes people call what we have called 'predicates' *atomic predicates*, and then use 'predicate' to refer to any formula containing free occurrences of variables. We will call a formula with one or more free variables an *open* formula

Syntax and semantics

- We have so far been interested in the *syntax* of $\mathcal{L}$
- Syntax concerns the signs of a language
    - The vocabulary
    - The construction rules
    - The inference rules
- Now it is time to move to what we are really interested in: the semantics of $\mathcal{L}$
- Semantics concerns the interpretation of a language
- An interpretation of a language determines which sentences in that language are true

What is an interpretation?

- We start by picking out a domain of discourse (or better yet, a domain of quantification). This domain is a set. It can be any set we like, however natural or gerrymandered. It could be the set of people, the set of numbers, the set of Queen Elizabeth II and Freddy Mercury, or whatever
- To each individual constant of $\mathcal{L}$ we assign an object as its referent
- To each sentence letter we assign one of the truth-values True or False
- To each one-place predicate we assign a "property"; or more accurately, a subset of our domain
- To each *n*-ary predicate we assign an "*n*-place relation"; or more accurately, a set of ordered *n*-tuples of objects from the domain. So for example, a dyadic predicate will be assigned a set of ordered pairs

An interpretation as an ordered pair

- In more formal terms, an interpretation, $\mathcal{I}$, is itself an ordered pair of a set D and an assignment v; i.e. $\mathcal{I} = \langle D, v \rangle$
- The set D is the domain of quantification
- The assignment v maps names to their objects, sentence letters to truth-values and predicates to "properties/relations"
- For example we might have
  - $D = \{\text{Joe, Rachel, Brian}\}$
  - $v(\text{'}a_1\text{'}) = \text{Joe}$
  - $v(\text{'}a_2\text{'}) = \text{Rachel}$
  - $v(\text{'}a_3\text{'}) = \text{Brian}$
  - $v(\text{'}F^1_1\text{'}) = \{x : x \text{ is a man}\} = \{\text{Brian}\}$
  - $v(\text{'}P\text{'}) = \text{True}$

There are many interpretations

- It must be emphasised now that there are many, *many* interpretations we can give for $\mathcal{L}$
- We can use *any* set as a domain, and we can use *any* assignment we like
- Strictly then, we shouldn't talk about truth and falsity *simpliciter*. We should talk about truth and falsity *on a given interpretation*

Semantics for the quantifier-free fragment of $\mathcal{L}$

- Once we have given an interpretation of $\mathcal{L}$, $\mathcal{I}$, we can give rules for deciding which sentence of $\mathcal{L}$ are true on $\mathcal{I}$
- But rather than considering all the sentences of $\mathcal{L}$, let's just now focus on the quantifier-free sentences

Let $\mathcal{I}$ be any interpretation of $\mathcal{L}$ and 'ϕ' any sentence of $\mathcal{L}$

1. If 'ϕ' is a sentence letter, then 'ϕ' is true on $\mathcal{I}$ iff $\mathcal{I}$ assigns True to 'ϕ'
2. If 'ϕ' is atomic and is not a sentence letter, then 'ϕ' is true under $\mathcal{I}$ iff the objects that $\mathcal{I}$ assigns to the individuals constants in 'ϕ' are related (when taken in the order in which the constants appear in 'ϕ') by the relation that $\mathcal{I}$ assigns to the predicate in 'ϕ'
3. If 'ϕ' $=$ '$\neg\psi$', then 'ϕ' is true on $\mathcal{I}$ iff 'ψ' is not true on $\mathcal{I}$
4. If 'ϕ' $=$ '$(\psi \vee \chi)$', then 'ϕ' is true on $\mathcal{I}$ iff 'ψ' is true on $\mathcal{I}$ or 'χ' is true on $\mathcal{I}$ or both
5. If 'ϕ' $=$ '$\psi \wedge \chi$', then 'ϕ' is true on $\mathcal{I}$ iff 'ψ' is true on $\mathcal{I}$ and 'χ' is true on $\mathcal{I}$
6. If 'ϕ' $=$ '$(\psi \supset \chi)$', then 'ϕ is true on $\mathcal{I}$ iff 'ψ' is not true on $\mathcal{I}$ or 'χ' is true on $\mathcal{I}$ or both
7. 'ϕ' is false on $\mathcal{I}$ iff 'ϕ' is not true on $\mathcal{I}$

Recursive definitions

- This definition of truth for the quantifier-free fragment of $\mathcal{L}$ starts by assigning truth to atomic sentences. Then we define truth for sentences built out of atomic sentences, and then sentences built out of those sentences, and so on
- Every quantifier-free sentence of $\mathcal{L}$ is either an atomic sentence, or is built up from atomic sentences by finitely many applications of the logical connectives. So this definition will cover every quantifier-free sentence
- This kind of definition is called *recursive*
- Of course, we cannot do quite the same thing for the quantifiers. '$\forall x_1 F^1_1 x_1$', for example, is not built out of atomic sentences

A reminder of the intuitive idea for the quantifiers

- Remember that the intuitive idea was that when we come across a sentence of the form
  - $\forall x \phi x$

  we replace the variable with a new name 'b', giving us
  - ϕb

  If this new sentence is true on all the different ways of reading the name 'b', then '$\forall x \phi x$' is true, otherwise '$\forall x \phi x$' is false
- Similarly,
  - $\exists x \phi x$

  is true iff
  - ϕb

  is true on at least one way of reading the name 'b'

β-variants

- The first step in making this intuitive idea is to introduce the idea of a β-*variant*
- An interpretation $\mathcal{J}$ is a β-variant of an interpretation $\mathcal{I}$ iff $\mathcal{J}$ and $\mathcal{I}$ differ, if at all, only in what they assign to the individual constant 'β'
- For example, if we take 'a_1' to be the individual constant 'β',
    - $D = \{\text{Joe, Rachel, Brian}\}$
    - $v(\text{'}a_1\text{'}) = \text{Joe}$
    - $v(\text{'}F_1^1\text{'}) = \{x : x \text{ is a man}\} = \{\text{Brian}\}$

  is a β-variant of
    - $D = \{\text{Joe, Rachel, Brian}\}$
    - $v(\text{'}a_1\text{'}) = \text{Rachel}$
    - $v(\text{'}F_1^1\text{'}) = \{x : x \text{ is a man}\} = \{\text{Brian}\}$
- A couple of facts about β-variants
    - Every interpretation is a β-variant of itself
    - If $\mathcal{J}$ is a β-variant of $\mathcal{I}$, then $\mathcal{J}$ has the same domain as $\mathcal{I}$

Some last tweaks

- Before we can finally give a semantics for the quantifiers, we need to make one last tweak
- On our intuitive idea for dealing with
  - $\forall x \phi x$

  we introduce a *new* name 'b'
- Rather than introducing a new name, it is standard to use the first name which does not appear in 'ϕ'. (We order the names via their numerical subscripts)
- For ease, we write the result of replacing every occurrence of 'x' in 'ϕ' with the term 'β' as '$\phi[x/\beta]$'. So '$F^1_1 x_1[x_1/a_1]$' is '$F^1_1 a_1$'

A semantics for the quantifiers

Let $\mathcal{I}$ be any interpretation of $\mathcal{L}$, 'ϕ' any sentence of $\mathcal{L}$, 'α' any variable of $\mathcal{L}$ and 'β' the first individual constant not to occur in 'ϕ'

8 If 'ϕ' $=$ '$\forall\alpha\psi$', then 'ϕ' is true on $\mathcal{I}$ iff '$\psi[\alpha/\beta]$' is true on every β-variant of $\mathcal{I}$

9 If 'ϕ' $=$ '$\exists\alpha\psi$', then 'ϕ' is true on $\mathcal{I}$ iff '$\phi[\alpha/\beta]$' is true on at least one β-variant of $\mathcal{I}$

10 Nothing else is true on $\mathcal{I}$ or false on $\mathcal{I}$

An example

- Let $\mathcal{I}$ be
  - $D = \{\text{Joe, Rachel, Brian}\}$ (where Joe $\neq$ Rachel, Joe $\neq$ Brian and Rachel $\neq$ Brian)
  - $v(\text{'}a_1\text{'}) = \text{Joe}$
  - $v(\text{'}a_2\text{'}) = \text{Rachel}$
  - $v(\text{'}a_3\text{'}) = \text{Brian}$
  - $v(\text{'}F_1^1\text{'}) = \{x : x \text{ is a man}\} = \{\text{Brian}\}$
- Now take the sentence '$\forall x_1 F_1^1 x_1$'. Is it true?
- Well, we need to first replace 'x_1' with the first name which does not occur in '$F_1^1 x_1$'. This gives us '$F_1^1 a_1$'. Then we need to look at all the a_1-variants of $\mathcal{I}$; if they are all true then '$\forall x_1 F_1^1 x_1$' is true, otherwise '$\forall x_1 F_1^1 x_1$' is false
- Well, $\mathcal{I}$ is an a_1-variant of $\mathcal{I}$. And by 2, '$F_1^1 a_1$' is false on $\mathcal{I}$: Joe is not a member of $\{\text{Brian}\}$
- So, '$\forall x_1 F_1^1 x_1$' is false on $\mathcal{I}$

Another example

- Keeping $\mathcal{I}$ the same, now take the sentence '$\exists x_1 F^1_1 x_1$'. Is this sentence true?
- Again, we need to first replace 'x_1' with the first name which does not occur in '$F^1_1 x_1$', giving us '$F^1_1 a_1$'. Then we need to look at all the a_1-variants of $\mathcal{I}$; if at least one of them is true then '$\exists x_1 F^1_1 x_1$' is true, otherwise '$\exists x_1 F^1_1 x_1$' is false
- Well, here's one a_1-variant of $\mathcal{I}$, $\mathcal{J}$:
    - $D = \{$Joe, Rachel, Brian$\}$ (where Joe $\neq$ Rachel, Joe $\neq$ Brian and Rachel $\neq$ Brian)
    - v('a_1') $=$ Brian
    - v('a_2') $=$ Rachel
    - v('a_3') $=$ Brian
    - v('F^1_1') $= \{x : x$ is a man$\} = \{$Brian$\}$
- By 2, '$F^1_1 a_1$' is true on $\mathcal{J}$: Brian is a member of $\{$Brian$\}$
- So, '$\exists x_1 F^1_1 x_1$' is true on $\mathcal{I}$

Tarski's idea

- Recall that when we are giving semantics for quantifier-free sentences, all we need to do is assign truth-values to the atomic sentences and then define truth of non-atomic sentences in terms of the truth-values of atomic ones
- Also recall that we could not use that method when we moved to quantifiers. '$\forall x_1 F^1_1 x_1$' does not have any atomic sentences as parts
- Tarski's idea was to use a notion which is to predicates what truth is to sentences, and then define truth for quantifier sentences in terms of that notion
- The notion that Tarski appealed to was *satisfaction*. (Sometimes people use the converse relation *x is true of y*)
- Roughly, an object *a* satisfies the predicate '*x* is wise' iff *a* is wise. (Equivalently, '*x* is wise' is true of *a* iff *a* is wise)

Satisfaction and sequences

- As well as talking about single objects satisfying monadic predicates, we might want to talk about a pair of objects taken in a particular order satisfying a dyadic predicate. For example *a* and *b* in that order satisfy the predicate '*x* is older than *y*' iff *a* is older than *b*
- Equally, we want to talk about three objects in a particular order satisfying three-place predicates, and so on for any number
- To avoid complications, Tarski talked about objects infinite sequences of objects satisfying a predicate

Sequences

- What is an infinite sequence of objects? Think back to your 1A set theory lectures. There you were introduced to the idea of *ordered pairs*: $\langle a, b \rangle$. These were just like sets, except the order matters, and the same object can occur in an ordered pair more than once
- As well as ordered pairs, we can have ordered triples, ordered quadruples, etc. An *infinite sequence* of objects is one of these ordered sets, but with infinitely many members
- Note that just as with sets, there is no requirement that infinite sequences be "natural". $\langle 1, 2, 3, 4 ... \rangle$ and $\langle$3, the Queen of England, 1, Freddy Mercury...$\rangle$ are both sequences
- The rough idea, then, is to say that a sequence s satisfies a predicate '$F_1^n x_1, x_2 ... x_n$' iff the first member of s, the second member of s... and the nth members of s in that order are related by the relation 'F_1^n' stands for

Satisfaction of predicates

- Let $\mathcal{I}$ be an interpretation of $\mathcal{L}$, s a sequence of objects from the domain of $\mathcal{I}$, and 'ϕ' an n-ary predicate followed by n individual symbols
- $\mathcal{I}$ assigns to every variable 'x_n' in 'ϕ' the nth member of s. (Remember that $\mathcal{I}$ already assigns objects to the names of $\mathcal{L}$)
- For example, if 'ϕ' is '$F^1_1 x_4$' and s is $\langle$Joe, Rachel, Brian, Rachel...$\rangle$, then $\mathcal{I}$ assigns Rachel to 'x_4'
- We then say that s satisfies 'ϕ' on $\mathcal{I}$ iff the objects that $\mathcal{I}$ assigns to the individual symbols in 'ϕ' are related (in the right order) by the relation that $\mathcal{I}$ assigns to the predicate in 'ϕ'

Tarskian semantics for the quantifier-free fragment of $\mathcal{L}$

Let $\mathcal{I}$ be any interpretation of $\mathcal{L}$, 'ϕ' any wff of $\mathcal{L}$, and *s* be any sequence of objects from the domain of $\mathcal{I}$

1. If 'ϕ' is a sentence letter then *s* satisfies 'ϕ' on $\mathcal{I}$ iff $\mathcal{I}$ assigns True to 'ϕ'
2. If 'ϕ' is an *n*-ary predicate followed by *n* individual constants, then *s* satisfies 'ϕ' on $\mathcal{I}$ iff the objects that $\mathcal{I}$ assigns to the individual symbols in 'ϕ' are related by the relation that $\mathcal{I}$ assigns to the predicate in 'ϕ'
3. If 'ϕ' = '$\neg\psi$', then *s* satisfies 'ϕ' on $\mathcal{I}$ iff *s* does not satisfy 'ψ' on $\mathcal{I}$
4. If 'ϕ' = '$(\psi \vee \chi)$', then *s* satisfies 'ϕ' on $\mathcal{I}$ iff *s* satisfies 'ψ' on $\mathcal{I}$ or *s* satisfies 'χ' on $\mathcal{I}$ or both
5. If 'ϕ' = '$\psi \wedge \chi$', then *s* satisfies 'ϕ' on $\mathcal{I}$ iff *s* satisfies 'ψ' on $\mathcal{I}$ and *s* satisfies 'χ' on $\mathcal{I}$
6. If 'ϕ' = '$(\psi \supset \chi)$', then *s* satisfies 'ϕ on $\mathcal{I}$ iff *s* satisfies 'ψ' on $\mathcal{I}$ or *s* satisfies 'χ' on $\mathcal{I}$ or both

Satisfaction and quantifiers

Let $\mathcal{I}$ be any interpretation of $\mathcal{L}$, 'ϕ' any wff of $\mathcal{L}$, and s be any sequence of objects from the domain of $\mathcal{I}$

7 If 'ϕ' = '$\forall x_n \psi$', then s satisfies 'ϕ' on $\mathcal{I}$ iff every sequence s' that differs from s only at its nth place (if at all) satisfies 'ψ' on $\mathcal{I}$

8 If 'ϕ' = '$\exists x_n \psi$', then s satisfies 'ϕ' on $\mathcal{I}$ iff some sequence s' that differs from s only at its nth place (if at all) satisfies 'ψ' on $\mathcal{I}$

9 s satisfies nothing else on $\mathcal{I}$

An example

- Let our domain be {Joe, Rachel, Brian}, let $\mathcal{I}$ assign {Brian} to 'F_1^1', and let s be $\langle$Joe, Rachel, Brian, Rachel...$\rangle$
- By 7, s satisfies '$\forall x_1 F_1^1 x_1$' iff every sequence s' which differs from s at most at the first place satisfies '$F_1^1 x_1$'
- s is a sequence which differs from s at most at the first place. (s doesn't differ from s at all!)
- By 2, s satisfies '$F_1^1 x_1$' iff Joe is a member of {Brian}
- Therefore, s does not satisfy '$F_1^1 x_1$'
- Therefore, s does not satisfy '$\forall x_1 F_1^1 x_1$'

Another example

- By 8, s satisfies '$\exists x_1 F^1_1 x_1$' iff some sequence s' which differs from s at most at the first place satisfies '$F^1_1 x_1$'
- Let $s' = \langle$Brian, Rachel, Brian, Rachel...$\rangle$. s' is a sequence which differs from s at most at the first place
- By 2, s' satisfies '$F^1_1 x_1$' iff Brian is a member of $\{$Brian$\}$
- Therefore, s' satisfies '$F^1_1 x_1$'
- Therefore, s satisfies '$\exists x_1 F^1_1 x_1$'

From satisfaction to truth

- We have a recursive definition of satisfaction. Now we need to turn this into a definition of truth
- We can do this in either one of two ways
  - A sentence 'ϕ' of $\mathcal{L}$ is true on $\mathcal{I}$ iff some sequence satisfies 'ϕ' on $\mathcal{I}$
  - A sentence 'ϕ' of $\mathcal{L}$ is true on $\mathcal{I}$ iff every sequence satisfies 'ϕ' on $\mathcal{I}$
- These two definitions are demonstrably equivalent
- Whether an atomic sentence is satisfied by a given sequence has nothing to do with the members of that sequence; it is just about the assignments $\mathcal{I}$ makes to sentence letters, predicates and individual constants.
- Equally then, whether a sentence built up from atomic sentences alone is satisfied by a given sequence has nothing to do with the members of that sequence
- So if one sequence satisfies a quantifier-free sentence, every sequence does

From satisfaction to truth — universal quantification

- Assume a sequence s satisfies the sentence '$\forall x_n \psi$'. Suppose for *reductio* that some sequence t does not satisfy '$\forall x_n \psi$'. By 7 there is some sequence t' that differs from t (if at all) only at the nth place and does not satisfy 'ψ'. There will be some sequence s' which has the same object as t' at the nth place but is otherwise the same as s. '$\forall x_n \psi$' is a sentence, and so the only free variable in 'ψ' (if there is one at all) is 'x_n'. By 2, if t' does not satisfy 'ψ' then neither does s'. So by 7 s does not satisfy '$\forall x_n \psi$'. Contradiction

From satisfaction to truth — existential quantification

- Assume a sequence s satisfies the sentence '$\exists x_n \psi$'. Suppose for *reductio* that some sequence t does not satisfy '$\exists x_n \psi$'. By 8 no sequence that differs from t (if at all) only at the nth place satisfies 'ψ'. For any sequence s' that differs from s only at the nth place, there will be some sequence t' which is the same as t except it has the same object at its nth place as s'. '$\exists x_n \psi$' is a sentence, and so the only free variable in 'ψ' (if there is one at all) is 'x_n'. By 2 it follows that no sequence which differs from s only in its nth member satisfies 'ψ'. So by 8 s does not satisfy '$\exists x_n \psi$'. Contradiction