Quantification 2
Frege's treatment of the quantifiers

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Last time

- We saw the shortcomings of Aristotelian logic
- It forces everything into categorical form, and thus eliminates relational expressions
- As a result it is unable to deal with multiple generality. It cannot deal with obviously valid inferences like ‘Everyone loves Aristotle’ to ‘Everyone loves someone’

Today's lecture

Frege's analysis of sentences

Constructional history

Uniform explanations

Quantifier-variable notation

Functions

- Frege started by looking at functions in mathematics
- Take, for example, the function x^2
- When we feed that expression with an argument, it spits out a value. So, when we insert the argument 9 we get 9^2, i.e. 81; when we insert the argument 10 we get 10^2, i.e. 100; etc
- Just as we can divide 9^2 into the function x^2 and the argument 9, we can split up the term '9^2' into the functional part 'x^2' and the numeral '9'

The functional analysis of sentences

- Frege's big idea is to abandon the traditional analysis of sentences into subject, copula and predicate, and move to a functional analysis
- Take the sentence
    - Shergar is a horse
- Frege divided that sentence into what we now call a predicate, 'is a horse', and a singular term, 'Shergar'. Frege uses fancy Greek variables to mark where the singular term goes:
    - ξ is a horse

Incomplete expressions

- According to Frege, predicates or relational expressions (with however many argument places) are incomplete (or unsaturated) because they have a gap for singular terms
- Singular terms, on the other hand, are complete (or saturated). They are arguments to predicates, but never themselves take arguments

Concepts and relations

- Frege called the referents of singular terms 'objects'
- Should we also say that predicates and relational expressions refer?
- This is controversial: more to come in lecture 4. Frege said that predicates and relational expressions do refer. He called the referents of predicates 'concepts' and the referents of relational expressions 'relations'

How far does the analogy go?

- Frege treated predicates, e.g. 'ξ is a horse', like functional expressions, e.g. 'x^2'. But how far does the analogy go? Are predicates just a type of functional expression, or are they a different class of expression?
- A functional expression like 'x^2' becomes a singular term when we replace its variable with a singular term. For instance, if we plug '9' into 'x^2' we get a singular term '9^2', which refers to the number 9^2, i.e. 81
- A predicate like 'ξ is a horse' becomes a singular term when we replace its variable with a singular term. For example, if we plug '9' into 'ξ is a horse' we get the sentence 'Shergar is a horse'
- So the question 'Are predicates a type of functional expression?' becomes the question 'Is a sentence a kind of singular term?'

Frege's two answers

- The early Frege's answer to this question was 'No'. In that case predicates and functional expressions are two different types of incomplete expression
- The later Frege's answer was 'Yes: sentences are singular terms referring to truth-values, the True and the False'. In that case predicates are literally functional expressions
- It is generally accepted that Frege was wrong to classify sentences as singular terms

Use and mention

- It is very important that we carefully distinguish the realm of language from the realm of reference
- At the level of language we have singular terms like ‘Shergar’ and predicates like ‘ξ is a horse’
- At the level of the world we have objects like Shergar and concepts like ξ is a horse

Back to quantifiers

- What does all of this have to do with quantifiers?
- Frege's analysis of sentences gives us the means to understand the internal structure of quantified sentences
- In order to give a correct account of the inferential powers of a quantified statement and its truth-conditions, we must understand its *constructional history*
- In other words, we must understand how a predicate to which a quantifier can then be prefixed is formed

Forming predicates from atomic sentences

- Take the atomic sentence
  - Odysseus misses Penelope
- We can form predicates and relational expressions from this sentence by omitting singular terms
  - Odysseus misses ξ
  - ξ misses Penelope
  - ξ misses ζ
- If we had started with 'Odysseus misses Odysseus' instead of 'Odysseus misses Penelope', then we could replace *both* occurrences of 'Odysseus' with *one* variable, giving us
  - ξ misses ξ

The order of quantifiers

- Now that we understand how predicates are formed from atomic sentences, we can form quantified sentences from them
- Consider again the sentence ‘Odysseus misses Penelope’
- We can omit the name ‘Penelope’, giving us the predicate ‘Odysseus misses ξ’
- And now we can insert a sign of generality into the gap, for example ‘Odysseus misses somebody’
- We can then omit the proper name ‘Odysseus’, giving us the new (complex) predicate ‘ξ misses somebody’
- Again we can replace the variable with a quantifier, yielding for example ‘Everybody misses somebody’

The linear approach

- We have been talking about quantified sentences being constructed in stages, that such sentences have a constructional history. But what is this talk of constructional history good for?
- To answer this question, we must look at the obvious alternative to thinking in terms of constructional histories
- Sentences are linearly ordered words on a page. The most natural way to account for the structure of sentences is to take them to have been built out of their component words arranged in linear order (Dummett, *Frege: Philosophy of Language*, edition 2, p.9)
- We will call this the *linear approach*
- According to the linear approach, 'Everyone misses someone' is thought to be constructed simultaneously out of its three components in the same way that 'Odysseus misses Penelope' is

The problem with the linear approach

- The problem with the linear approach is that it fails to provide a decent explanation of quantifiers that lie within the scope of the quantifiers
- Take the sentences 'Everyone misses someone' and 'Odysseus misses someone'
- On the face of it, the existential quantifier does not make the same contribution to the truth-conditions of these sentences. In the first case 'someone' does not indicate the presence of some *one* person; in the second case it does
- It might therefore be tempting to attribute different semantic roles to quantifiers depending on whether or not they appear in the scope of other quantifiers. This is exactly what the medievals did with their supposition theory
- But this precludes a uniform explanation of the contribution a quantifier makes to the truth-conditions of a sentence

The advantage of the constructional history approach

- The beauty of Frege's approach is that it allows us to give a uniform treatment of quantifiers
- The constructional history approach allows us to explain sentences involving multiple generality (no matter how complex) on the model of sentences involving only one quantifier

Sentences with one quantifier

- Very roughly, the sentence
    - Everyone sang

  is true iff all of
    - Joe sang
    - Rachel sang
    - Brian sang
    - ...

  are true
- Very roughly, the sentence
    - Someone sang

  is true iff at least one of
    - Joe sang
    - Rachel sang
    - Brian sang
    - ...

  is true

Sentences with more than one quantifier

- We can now analyse 'Everyone misses someone' in accordance with its constructional history
- 'Everyone misses someone' was obtained from the predicate 'ξ misses someone'. Therefore, it is true just in case all of
  - Joe misses someone
  - Rachel misses someone
  - Brian misses someone
  - ...

  are true
- 'Joe misses someone' was obtained from the predicate 'Joe misses ξ'. Therefore it is true just in case at least one of
  - Joe misses Lucy
  - Joe misses Bill
  - ...

  is true
- The same goes for 'Rachel misses someone', 'Brian misses someone', etc.

The key point

- The existential quantifier is given a uniform treatment whether or not it lies within the scope of another quantifier. The same goes for the universal quantifier
- This is because by the time we reach the quantifier 'someone' in the constructional history of 'Everyone misses someone', we have already stripped away the quantifier 'Everyone'

The problem of scope

- We most naturally hear the sentence 'Everyone misses someone' as the result of plugging a universal quantifier into 'ξ misses someone'
- But we might also want to express the sentence resulting from plugging the existential quantifier into 'Everyone misses ξ'. This sentence would be true if at least one of
  - Everyone misses Lucy
  - Everyone misses Bill
  - ...

  is true
- But we have a problem. If we just replace the variable in 'Everyone misses ξ' with 'someone', then we just get 'Everyone misses someone'. How can we tell this apart from the sentence we get by replacing the variable in 'ξ misses someone' with 'everyone'?
- In technical terms, the problem is telling which quantifier has the wider scope: 'everyone' or 'someone'?

Natural language solutions

- Context — expect our audience to be able to pick up what we mean
- Emphasis
  - *Everyone* misses someone — the result of replacing the variable in 'ξ misses someone' with 'everyone'
  - Everyone misses some*one* — the result of replacing the variable in 'Everyone misses ξ' with 'someone'
- Active versus passive
  - Everyone misses someone — the result of replacing the variable in 'ξ misses someone' with 'everyone'
  - Someone is missed by everyone — the result of replacing the variable in 'Everyone misses ξ' with 'someone

This solution relies on the rule that the order of construction corresponds to the inverse order in which the signs of generality occur in the sentence, i.e. if 'everyone' precedes 'someone' on the page, it is introduced at a later step in the construction

Problems with the natural language solutions

- Context is often an imprecise tool. And anyway, we often don't want the meaning of our sentences to be tied to a particular context (or tied any closer than it has to be!)
- Emphasis is sometimes ineffective in getting our point across. It is also hard to replicate in the written word. And anyway, how are we to apply this method to cases with three or more quantifiers, e.g. 'Everyone loves someone who loves everyone'?
- Active and passive voice relies on a substantial redundancy in our language. For every sentence featuring n signs we need the syntactic resources to produce each of the possible permutations (and hence alternative orders of stage-by-stage construction)

The quantifier-variable notation

- It is easy to see that an orderly theory of quantification requires us to introduce notational means which allow us to dispense with such verbal contortions
- Frege's solution (which we still use today) was to prefix the quantifiers to the left of the sentence
  - $\forall x \exists y (x \text{ misses } y)$ — Everyone misses someone
  - $\exists y \forall x (x \text{ misses } y)$ — Someone is missed by everyone
- The further left the quantifier appears, the later it appears in the constructional history; in other words, the further left the quantifier appears, the wider its scope

Cross-referencing

- Shifting to Frege's quantifier-variable notation also allows us to deal neatly with cross-referencing
- Sometimes we want to say several things of one object
- In natural language we do this with the help of pronouns
  - Someone killed himself
- But when we come to sentences with lots of cross-referencing, this method soon becomes confusing
  - Someone despises everyone who kills himself, and he loves everyone who kills someone other than himself
- The quantifier-variable notation allows us to unproblematically express sentences with any degree of cross-referencing. So, one of the things the difficult sentence above might mean is
  - $\exists x \forall y \forall z \forall w$((if y kills y then x despises y) and ((z kills w and $w \neq z$) then x loves z))