Paradoxes
Lecture Six

Vagueness: The Sorites Paradox

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Vagueness: The Sorites Paradox

Last Week

Supervaluationism

Supervaluationism and the Sorites Paradoxes

An Objection to Supervaluationism

Many-Valued Logic

Many-Valued Logic and the Sorites Paradoxes

Objections to Many-Valued Logic

Summary
The Paradox of the Heap

- 1 grain of sand does not make a heap
The Paradox of the Heap

- If 1 grain of sand does not make a heap, then 2 grains of sand do not make a heap
The Paradox of the Heap

- If 2 grains of sand do not make a heap, then 3 grains of sand do not make a heap
The Paradox of the Heap

999,999,997 steps later...
The Paradox of the Heap

- If 999,999,999 grains of sand do not make a heap, then 1,000,000,000 grains of sand do not make a heap!
Vagueness

- The Paradox of the Heap exploits the fact that *heap* is a vague concept
- We can set up a Sorites paradox for any vague concept
  - Bald
  - Tall
  - Table (!?)
General Form A

- $\phi(0)$
- $\forall n (\phi(n) \rightarrow \phi(n + 1))$ (QP)
- $\therefore \forall n \phi(n)$
General Form B

- $\phi(0)$
- $\phi(0) \rightarrow \phi(1)$
- $\phi(1) \rightarrow \phi(2)$
- ...
- $\phi(m - 1) \rightarrow \phi(m)$
- $\therefore \phi(m)$

- $m$ is a sufficiently large number that it is obviously false to say $\phi(m)$
Vague Concepts as Faulty Concepts

- One way of responding to the Sorites paradoxes is to say that vague concepts are inherently flawed, and so cannot be used to describe the world
- This was Unger’s view: there are no heaps or tables, no body is bald or tall
- Frege held a similar view, but went even further: vague concepts are so faulty that they cannot even be meaningfully applied at all!
Epistemicism

- Another way is by denying one of the conditionals of this form:
  \[ \phi(n) \rightarrow \phi(n+1) \]
- If we continue to accept classical logic, denying this conditional amounts to asserting:
  \[ \phi(n) \land \neg \phi(n+1) \]
- In other words, it involves asserting that there is a sharp cut off line between \( \phi \) and \( \neg \phi \)
- This might sound absurd: what is the cut off between being bald and being not-bald?
- According to Williamson’s **epistemicism**, there is an answer to that question; it’s just that we **cannot know** the answer
Comparing these two Responses

- Unger’s “vague concepts are faulty concepts” view is in many ways similar to Williamson’s epistemicism.
- They both deal with the Sorites paradoxes without complicating our semantic theories or our logical rules.
- But in a certain sense, they both refuse to take vagueness seriously:
  - Unger: vague concepts are just broken concepts
  - Williamson: to say that a concept is vague is just to say that we do not know where its boundary lies
- This week, we will look at two responses to the Sorites paradoxes which take vagueness more seriously.
Paradoxes (6): The Sorites Paradox

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Vagueness: The Sorites Paradox

Last Week

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Summary
Vagueness as Semantic Imprecision

- The way we use the word ‘bald’ makes it clearly right to call some people bald.
- The way we use the word ‘bald’ makes it clearly right to call some people not-bald.
- But there are **borderline cases**, people who are not rightly called bald and who are not rightly called not-bald.
- This is because the way we use the word ‘bald’ just isn’t precise enough to settle in each and every case whether someone is bald or whether they are not-bald.
Vagueness as Semantic Imprecision

- The way we use the word ‘bald’ makes it right to call A and B bald.
Vagueness as Semantic Imprecision

- And the way we use the word ‘bald’ makes it right to call D not-bald
Vagueness as Semantic Imprecision

- But the way we use the word ‘bald’ does not make it right to call C bald or to call C not-bald
Vagueness as Semantic Imprecision

- The way we use the word ‘bald’ is too imprecise to do that
Making Vague Concepts more Precise

- But we could make our use of the word ‘bald’ more precise, if we liked
Making Vague Concepts more Precise

- We could make stipulate that from now on, C will count as bald
Making Vague Concepts more Precise

- Or we could make stipulate that from now on, C will count as not-bald
Precisifications

- These ways of making ‘bald’ more precise are known as **precisifications** of ‘bald’
- These precisifications assign a precise extension to ‘bald’
  - The extension of a predicate is the set of things which that predicate applies to
Precisifications

- Every admissible precisification of ‘bald’ needs to conform to the following rules:
  - If $x$ is clearly bald, then $x$ must be in the extension of ‘bald’
  - If $x$ is clearly not-bald, then $x$ must not be in the extension of ‘bald’
  - If $x$ is in the extension of ‘bald’ and $y$ is more bald (i.e. has fewer hairs) than $x$, then $y$ must be in the extension of ‘bald’ too
  - If $x$ is not in the extension of ‘bald’ and $x$ is more bald than $y$, then $y$ must not be in the extension of ‘bald’ either

- There will be lots of different precisifications of ‘bald’ which conform to all of these rules
Supervaluationism

- According to **supervaluationism**, when we want to figure out whether a claim involving the word ‘bald’ is true or false, we need to look at all these different precisifications of ‘bald’
- ‘Paul is bald’ is true if and only if it is true on all admissible precisifications of ‘bald’
- ‘Paul is bald’ is false is true if and only if it is false on all admissible precisifications of bald
  - ‘P’ is false iff ‘not-P’ is true
Supervaluationism

- And of course, what goes for the predicate ‘bald’ goes for all vague predicates.
- A precisification of a whole language, $L$, assigns precise extensions to all the predicates in $L$, obeying the rules on admissible precisifications given earlier:
  - If $x$ is clearly $F$, then $x$ must be in the extension of ‘$F$’
  - If $x$ is clearly not-$F$, then $x$ must not be in the extension of ‘$F$’
  - If $x$ is in the extension of ‘$F$’ and $y$ is more $F$ than $x$, then $y$ must be in the extension of ‘$F$’ too
  - If $x$ is not in the extension of ‘$F$’ and $x$ is more $F$ than $y$, then $y$ must not be in the extension of ‘$F$’ either
Supervaluationism

- $A$ is true if and only if $A$ is true on every admissible precisification of the language

- $A$ is false if and only if $A$ is false on every admissible precisification of the language

- Supervaluationism has been developed and defended by many philosophers, but see especially Fine’s (1975) and Keefe’s (2000)
Bivalence versus LEM

- Supervaluationism gives up on bivalence, according to which every sentence is either true or false (but not both)
- Suppose that Paul is neither clearly bald, nor clearly not-bald
- Now consider the sentence ‘Paul is bald’
- This sentence is not true
  - Since Paul is not clearly bald, there will be some precisification of ‘bald’ on which ‘Paul is bald’ is not true
- The sentence is not false
  - Since Paul is not clearly not-bald, there will be some precisification of ‘bald’ on which ‘Paul is bald’ is true, and hence not false
- Thus ‘Paul is bald’ is neither true nor false
Bivalence versus LEM

- But interestingly, supervaluationism does not need to give up on the **Law of Excluded Middle** (LEM):
  - \( P \) or \( \neg P \)
- ‘Paul is bald or not-(Paul is bald)’ is true on every precisification!
Bivalence versus LEM

- Consider the precisifications which do put Paul in the extension of ‘bald’
Bivalence versus LEM

- ‘Paul is bald’ is true on these precisifications...
Bivalence versus LEM

...and thus so is ‘Paul is bald or not-(Paul is bald)’
Now consider the precisifications which do not put Paul in the extension of ‘bald’.
Bivalence versus LEM

- ‘not-(Paul is bald)’ is true on these precisifications...
Bivalence versus LEM

...and thus so is ‘Paul is bald or not-(Paul is bald)’
Therefore ‘Paul is bald or not-(Paul is bald)’ is true on all the admissible precisifications.
Bivalence versus LEM

- Thus, LEM is not undermined by supervaluationism
- This is good, since LEM is a classical law of logic
- And in fact, supervaluationism is compatible with all the classical laws of logic
Vagueness: The Sorites Paradox

Last Week

Supervaluationism

Supervaluationism and the Sorites Paradoxes

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Many-Valued Logic

Many-Valued Logic and the Sorites Paradoxes

Objections to Many-Valued Logic

Summary
Let’s start with Form B of the Sorites paradoxes:

A man with 0 hairs is bald
If a man with 0 hairs is bald, then a man with 1 hair is bald
If a man with 1 hair is bald, then a man with 2 hairs is bald
...
If a man with 9,999 hairs is bald, then a man with 10,000 hairs is bald
Therefore a man with 10,000 hairs is bald

According to supervaluationism, at least one of the conditional premises will not be true
Suppose that people with 23 hairs and people with 24 hairs are **borderline cases** of being bald:
- They are neither clearly bald, nor clearly not-bald

Is this sentence true: ‘If a man with 23 hairs is bald, then a man with 24 hairs is bald’?
- Since 23 and 24 are both borderline cases, there will be an admissible precisification of ‘bald’ which puts people with 23 hairs in the extension of ‘bald’, but not people with 24 hairs
- On this precisification, ‘If a man with 23 hairs is bald, then a man with 24 hairs is bald’ is not true

So this conditional is not true on every admissible precisification, and thus is not true full stop

Thus the Sorites paradoxes are premise-flawed, in the sense that they rely on untrue premises
Sharp Boundaries

• Earlier I said that in classical logic, denying this conditional:
  – A man with 23 hairs is bald → a man with 24 hairs is bald is equivalent to asserting this:
  – A man with 23 hairs is bald, and a man with 24 hairs is not bald
• I also said that supervaluationism obeys classical logic
• So doesn’t supervaluationism have to say that there is a sharp cut-off line between being bald and not-bald?
• No!
Neither True nor False

- When I said that denying ‘$P \rightarrow Q$’ is equivalent to asserting ‘$P \land \neg Q$’, I meant that calling ‘$P \rightarrow Q$’ false is equivalent to calling ‘$P \land \neg Q$’ true.
- But according to supervaluationism, this conditional is not true or false:
  - A man with 23 hairs is bald $\rightarrow$ a man with 24 hairs is bald.
- We have already seen why it is not true, and it is easy to see that it also is not true:
  - There will be some admissible precisification of ‘bald’ which puts people with 23 hairs and people with 24 hairs in the extension of ‘bald’.
  - On these precisifications, ‘If a man with 23 hairs is bald, then a man with 24 hairs is bald’ is true, not false.
Form A

• Now let’s look at Form A of the Sorites paradoxes:

  A man with 0 hairs is bald
  \( \forall n \) (If a man with \( n \) hairs is bald, then a man with \( n + 1 \) hairs is bald) (QP)
  Therefore \( \forall n \) (a man with \( n \) hairs is bald)

• According to supervaluationism, QP is not true
We have just seen that if 23 hairs and 24 hairs are borderline cases of ‘bald’, then there will be some admissible precisification of ‘bald’ on which the following is not true:
- A man with 23 hairs is bald → a man with 24 hairs is bald

But this conditional is an instance of QP for ‘bald’:
- \( \forall n (\text{a man with } n \text{ hairs is bald } \rightarrow \text{a man with } n + 1 \text{ hairs is bald}) \)

So QP cannot be true either
Sharp Boundaries (Again!)

- QP is not just not-true: it is false!
- On every precisification, there is a sharp line between people who are bald and people who are not
- Different precisifications draw this line in different places, but they all have one
- So QP will turn out to be false on every single precisification
- But in classical logic, saying that QP is bald is equivalent to saying that this is true:
  - \( \exists n (\text{a man with } n \text{ hairs is bald and a man with } n + 1 \text{ hairs is not bald}) \)
- This surely says that there is a sharp cut off between being bald and not somewhere, it just doesn’t tell us where
- So now haven't the supervaluationists ended up positing sharp boundaries after all?
Existential Generalisations in Supervaluationism

- We already saw that according to supervaluationism, **disjunctions** can be true even when both its disjuncts are not true:
  - ‘Paul is bald or not-(Paul is bald)’ is true
  - ‘Paul is bald’ is not true
  - ‘not-(Paul is bald)’ is not true

- In exactly the same way, according to supervaluationism, **existential generalisations** can be true even when the generalisation is not true of any particular thing
  - ‘\( \exists n (a\ \text{man with } n\ \text{hairs is bald and an}\ \text{man with } n+1\ \text{hairs is not bald}) \)’ is true
  - ‘A man with \( n\ \text{hairs is bald and a man with } n+1\ \text{hairs is not bald} \)’ is not true of any number \( n \)

- This can look a little bit weird, but it’s just how supervaluationism works
Vagueness: The Sorites Paradox

Last Week

Supervaluationism

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Summary
A Summary

- Supervaluationism is very attractive
  - It takes vagueness very seriously, as a real property of various predicates
  - It preserves classical logic, in the sense of preserving classically valid inferences and obeying classical laws of logic
- However, it does have some weird features
  - Disjunctions can be true even when neither of their disjuncts are
  - Existential generalisations can be true even when the generalisation is not true of anything
- Perhaps we can live with these oddities, but there is another objection
Higher-Order Vagueness

- Admissible precisifications of ‘bald’ must put everyone who is clearly bald into the extension of ‘bald’
- But who is clearly bald?
- There is no sharp cut off line between being clearly bald and being not clearly bald
- The predicate ‘clearly bald’ is just as vague as the predicate ‘bald’
- In fact, we could set up a Sorites paradox for clearly bald:
  - A man with 0 hairs is clearly bald; if a man with \( n \) hairs is clearly bald, then so is a man with \( n + 1 \) hairs; so a man with 10,000 hairs is clearly bald
Vague Semantics for Vague Languages

- It is not 100% clear how serious this problem is for supervaluationism.
- It is really bad news if you wanted to give a non-vague semantics for a vague language.
- But maybe we can accept that the semantics of a vague language has to be vague too?
- If they took this line, supervaluationists would just say that it is sometimes vague whether a precisification counts as admissible.
  - For example: if Paul is a borderline case of being clearly bald, then a precisification which does not put him in the extension of ‘bald’ is a borderline case of being admissible.
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Last Week

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Summary
It is completely true to say that A is bald
More or Less True

- It is completely false to say that D is bald
More or Less True

- It is not quite completely true to say that B is bald
More or Less True

- It is even less true to say that C is bald
Degrees of Truth

- Some logicians take this way of speaking, “more or less true” very seriously
- They say that there are many truth values other than true (1) and false (0)
- There are lots of different ways of trying to fill this idea out
- I am going to focus on one standard way, sometimes known as fuzzy logic
  - However, for a potentially more interesting alternative, see Edgington (1996)
Continuum-Many Truth Values

- There are *continuum*-many truth values, which are represented with real numbers between 0 and 1
- 0 represents complete falsehood, 1 represents complete truth, and the closer we get to 1, the closer we get to truth
- A bit of notation: \([A]\) is the truth-value of the sentence \(A\)
  - If \(A\) is completely true, then \([A] = 1\)
  - If \(A\) is completely false, then \([A] = 0\)
  - If \(A\) is somewhere in between, then \([A]\) is some number between 1 and 0, e.g. 0.75
A Fuzzy Semantics

\[
\begin{align*}
\neg A & = 1 - [A] \\
A \land B & = \min\{[A], [B]\} \\
A \lor B & = \max\{[A], [B]\} \\
A \rightarrow B & = 1 \text{ if } [B] \geq [A] \\
& \quad 1 - ([A] - [B]) \text{ otherwise} \\
\forall x A x & = \text{glb}\{[A^x/a] : \text{for all } a\} \\
\exists x A x & = \text{lub}\{[A^x/a] : \text{for all } a\}
\end{align*}
\]

(See Sainsbury and Williamson 1997, p. 476)
Validity in this Many-Valued Logic

- This semantics tells us how to decide which truth-values complex sentences get, in terms of the truth-values that the simple sentences get.
- But how do we decide whether an argument is valid in this logic?
- In classical logic, we say this:
  - An argument is valid iff there is no interpretation which makes the premises true and the conclusion false.
- But what should we say about interpretations that make the premises and/or conclusion somewhere between completely true and completely false?
Validity in this Many-Valued Logic

- On one standard account, we say this:
  - An argument is valid iff there is no interpretation which assigns the conclusion a lower truth value than the argument’s least true premise

- In classical logic, validity preserves truth
- In this fuzzy logic, validity preserves degree of truth
  - But see Edgington 1996 for an interesting alternative!
Vagueness: The Sorites Paradox

Last Week

Supervaluationism

Supervaluationism and the Sorites Paradoxes

An Objection to Supervaluationism

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Many-Valued Logic and the Sorites Paradoxes

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Form B

• Again, let’s start with Form B of the Sorites paradoxes

  A man with 0 hairs is bald
  If a man with 0 hairs is bald, then a man with 1 hair is bald
  If a man with 1 hair is bald, then a man with 2 hairs is bald
  ...
  If a man with 9,999 hairs is bald, then a man with 10,000 hairs is bald
  Therefore a man with 10,000 hairs is bald

• What truth-value do these premises get?
The Truth Values of the Premises and the Conclusion

- Surely \([\text{A man with 0 hairs is bald}] = 1\)
- And let’s say \([\text{A man with 1 hair is bald}] = 0.95\)
- In that case: \([\text{If a man with 0 hairs is bald then a man with 1 hair is bald}] = 1 - (1 - 0.95) = 0.95\)
- So at least one of the conditional premises is not completely true
- Does this mean that the paradox is premise-flawed?
- No: in this many-valued logic, it is still paradoxical that premises with very high truth values lead to a conclusion with a very low truth value
  - We can suppose that \([\text{A man with 10,000 hairs is bald}] = 0\)
Modus Ponens is Invalid

- The real solution to the Sorites paradoxes in this many-valued logic is to say that they are *fallacious*.
- This might come as a surprise, because the only rule of inference used in this paradox is *modus ponens*:
  - $A \rightarrow B; A; \therefore B$
- But in the fuzzy logic we have described, modus ponens really is invalid.
Modus Ponens is Invalid

- We said \([\text{A man with 1 hair is bald}] = 0.95\)
- Let’s also suppose that \([\text{A man with 2 hairs is bald}] = 0.9\)
- In that case: \([\text{If a man with 0 hairs is bald then a man with 2 hairs is bald}] = 1 - (0.95 - 0.9) = 0.95\)
- So applying modus ponens here would take us to a conclusion with a lower truth value than any of the premises
  - \([\text{If a man with 1 hair is bald then a man with 2 hairs is bald}] = 0.95\)
  - \([\text{A man with 1 hair is bald}] = 0.95\)
  - \([\text{A man with 2 hairs is bald}] = 0.9\)
- Modus ponens does not preserve degree of truth, and so is not valid
Form A

- Form A of the paradox is also invalid:
  A man with 0 hairs is bald
  $\forall n$ (If a man with $n$ hairs is bald, then a man with $n + 1$ hairs is bald) (QP)
  Therefore $\forall n$(a man with $n$ hairs is bald)

- But I will leave it as an exercise to the audience to figure out how!
Vagueness: The Sorites Paradox

Last Week

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Supervaluationism and the Sorites Paradoxes

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Summary
Modus Ponens is Invalid!

- We just saw that the many-valued logic above solves the Sorites paradoxes by rejecting modus ponens as invalid.
- But this is a pretty radical thing to do
  - Modus ponens is a very basic rule of inference!
- Admittedly, we don’t have to reject all uses of modus ponens.
- Modus ponens still works fine when applied to sentences that are completely true or completely false.
- But vagueness is everywhere, and if we represent vagueness with truth values between 1 and 0, then we will almost never be able to use modus ponens.
Odd Truth Values

• Suppose Paul is an exact borderline case of being bald, so that \([\text{Paul is bald}] = 0.5\)
• In that case: \([\text{Paul is not bald}] = 1 - [\text{Paul is bald}] = 1 - 0.5 = 0.5\)
• So \([\text{Paul is bald and Paul is not bald}] = 0.5\)
• But ‘Paul is bald and Paul is not bald’ is a contradiction!
• Shouldn’t \([\text{Paul is bald and Paul is not bald}] = 0\)?
Degrees of truth seem to be a great way of denying that there is a sharp cut off line between being bald and being not bald.

As people get more hairs, it becomes less and less true to say that they are bald.

But there is still a sharp cut off between people who are completely bald (bald to degree 1), and everyone else.

And there is also still a sharp cut off between people who are completely not bald (bald to degree 0), and everyone else.

But bald to degree 1 and bald to degree 0 seem just as vague as bald.
Vagueness: The Sorites Paradox

Last Week

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Summary
Not Taking Vagueness Seriously

- We have looked at four responses to the Sorites paradoxes
- Two of them didn’t take vagueness very seriously
  - Unger’s view that vague concepts are faulty concepts
  - Williamson’s epistemicism
- The benefit of these views is that they do not call for any modifications to logic or semantics
- The downside of these views is that they just seem somewhat incredible
  - Are there really no tables?
  - Is there really an unknowable sharp boundary between being bald and being not-bald?
Taking Vagueness Seriously

- We also looked at two responses which did take vagueness seriously
  - Supervaluationism
  - Many-valued logics (Fuzzy logic)

- The benefits of these views is they give full importance to the idea that vagueness is a real feature of our concepts and predicates

- But they are open to a number of objections, perhaps most importantly the problem of higher-order vagueness
What are we to do?

- I think it’s fair to say that it is not easy to pick between these options
- None of them are perfect!
- Good news (or perhaps bad): there are a whole lot more options to consider!
  - See the SEP article on the Sorites for some of them
Next Week

- Next week we will be looking at the Liar Paradox
- Required reading: *Paradoxes* ch. 6, §§6.2–6.8
References

- Unger, P (1979) ‘There are no ordinary things’, *Synthese* 41: 117–54