Paradoxes
Lecture Three

Zeno's Paradoxes

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Zeno's Paradoxes

Zeno of Elea

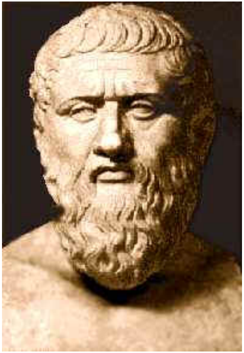
Achilles and the Tortoise

The Racetrack

The Arrow

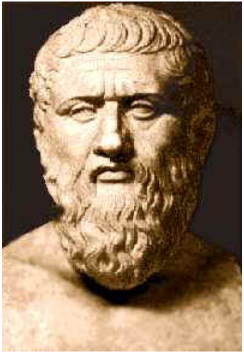
Zeno of Elea

- Born c. 490BCE
- Disciple of Parmenides
- Plato's *Parmenides* reports a visit of Zeno and Parmenides to Athens, where Socrates goes to hear Zeno read from his book of paradoxes



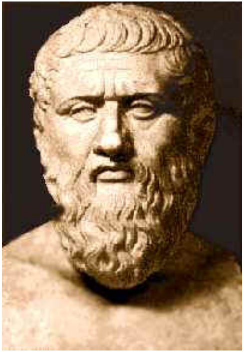
Zeno of Elea

- As Parmenides' disciple, he wanted to show that despite appearances to the contrary, reality is a **unified unchanging whole**
- His paradoxes were supposed to demonstrate the incoherence of our ordinary view of the world as made up of discrete, changing parts



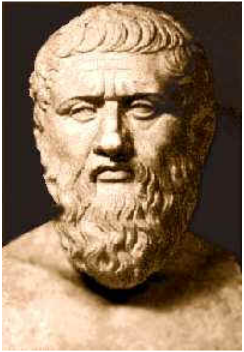
Zeno of Elea

- Very little survives of his writings
- We know about his paradoxes primarily through Aristotle's discussion of them in his *Physics*



Zeno's Paradoxes

- Achilles and the Tortoise (aka The Achilles)
- The Racetrack (aka The Dichotomy)
- The Arrow
- The Stadium



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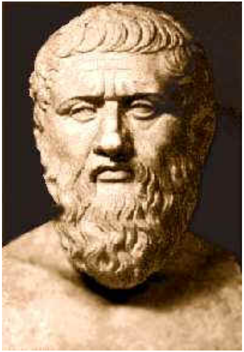


Why Study Zeno's Paradoxes?

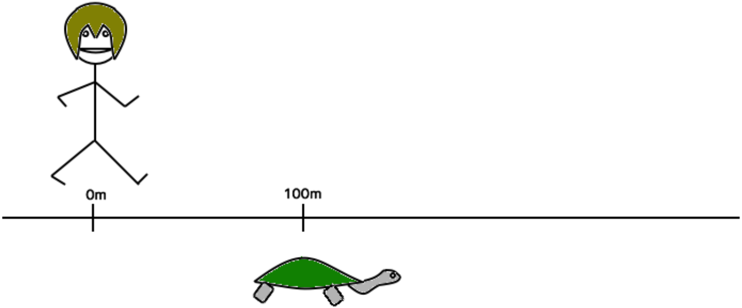
- All of these paradoxes are designed to convince you that motion is **impossible**
- This conclusion is **obviously absurd**, so why should we bother studying Zeno's arguments???
- Although we all know that there must be something wrong with Zeno's arguments, it is remarkably difficult to explain just what it is
- Whenever a philosopher thinks they have gotten to the bottom of the paradoxes, a new puzzle always seems to turn up

Zeno's Onion

> *Zeno's paradoxes have an onion-like quality; as one peels away outer layers by disposing of the more superficial difficulties, new and more profound problems are revealed.*
>
> *Salmon (1970) p. 43*

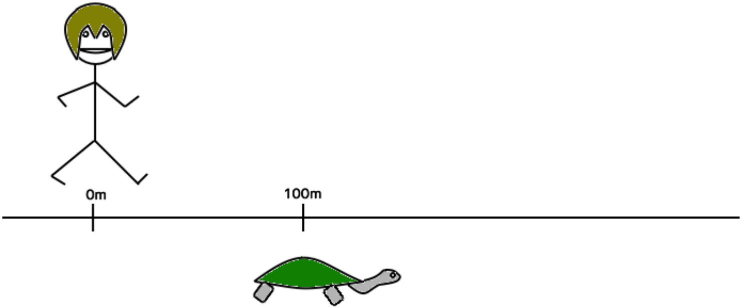
Zeno's Paradoxes

Achilles and the Tortoise



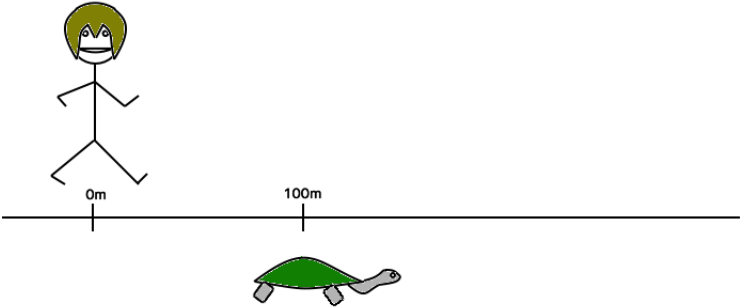
- Achilles and a tortoise are having a race

Achilles and the Tortoise



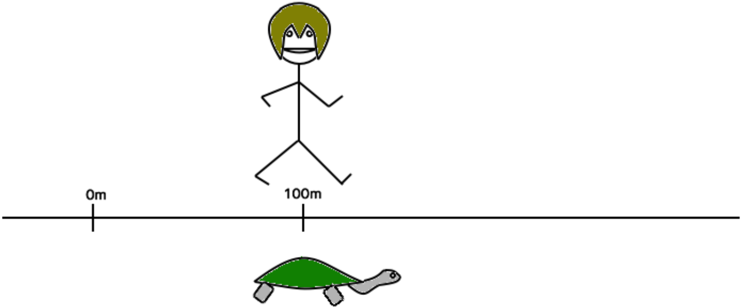
- Achilles is 10 times faster than the tortoise

Achilles and the Tortoise



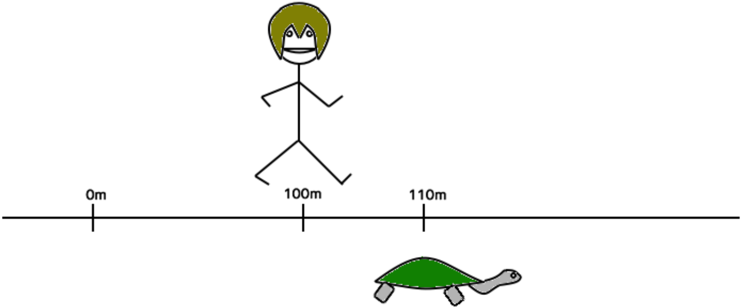
- So the tortoise is given a 100m headstart

Achilles and the Tortoise



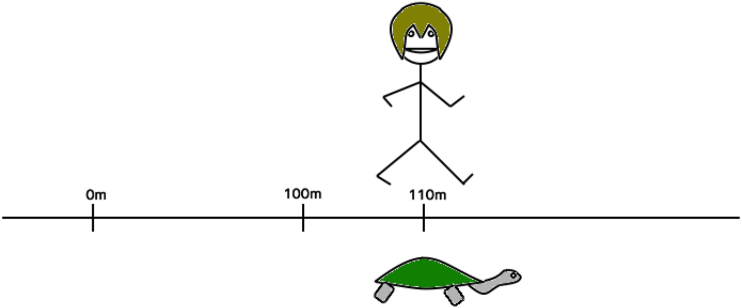
- Achilles quickly covers the 100m headstart

Achilles and the Tortoise



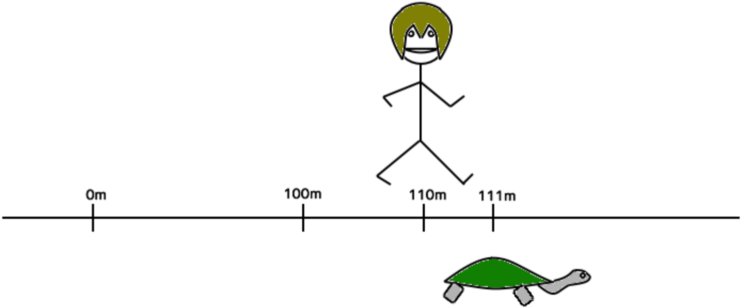
- But by that time, the tortoise has moved another 10m

Achilles and the Tortoise



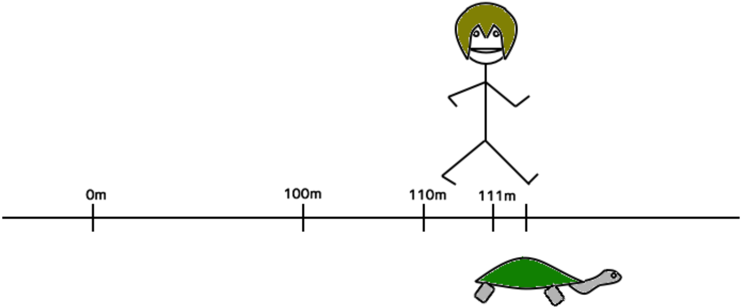
- Achilles quickly covers that 10m

Achilles and the Tortoise



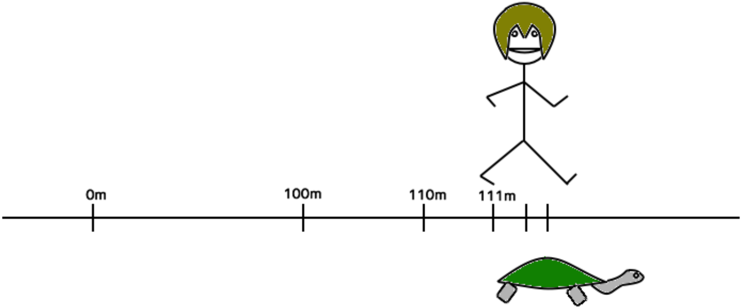
- But by that time, the tortoise has moved another 1m

Achilles and the Tortoise



- And on it goes...

Achilles and the Tortoise



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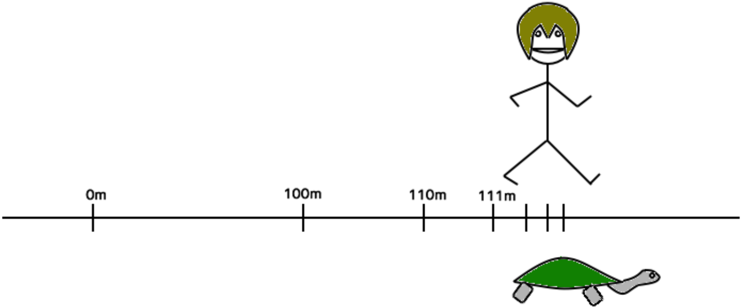
Achilles and the Tortoise



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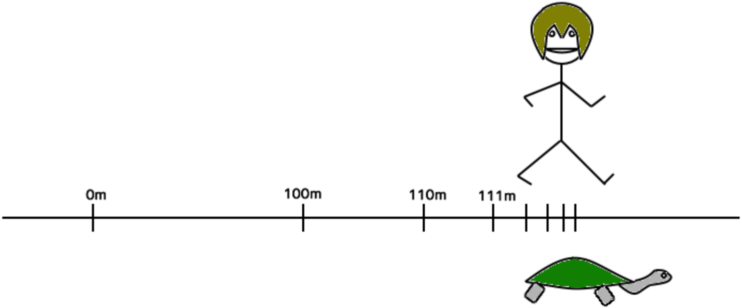
Achilles and the Tortoise



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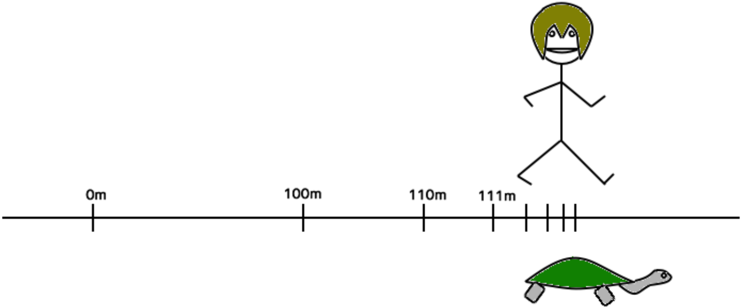
Achilles and the Tortoise



- It is impossible for Achilles to catch up with the tortoise

An Infinite Series of Distances

- When Achilles reaches the 100m mark, the tortoise has moved an extra 10m to the 110m mark
- When Achilles reaches the 110m mark, the tortoise has moved an extra 1m to the 111m mark
- When Achilles reaches the 111m mark, the tortoise has moved an extra 0.1m to the 111.1m mark
- ...

A Mathematical Solution to the Paradox

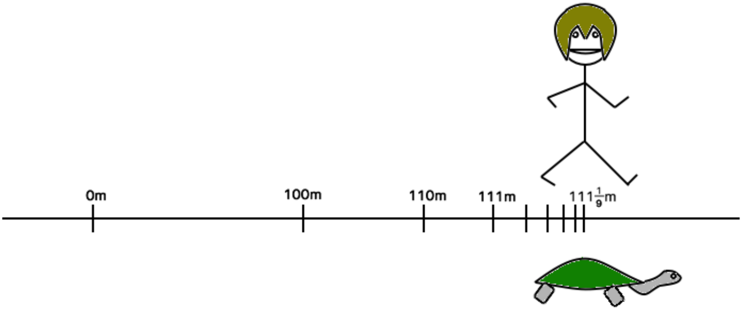
- So to catch up with the tortoise, Achilles must first cover 100m, then another 10m, then another 1m, then another 0.1m...
- In other words, he must cover this many meters:

$$100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + ... = 111.111...$$

- Is this possible? Yes!

$$111.111... = 111\frac{1}{9}$$

Achilles and the Tortoise



- Achilles catches up with the tortoise at $111\frac{1}{9}$m

Infinite Sums

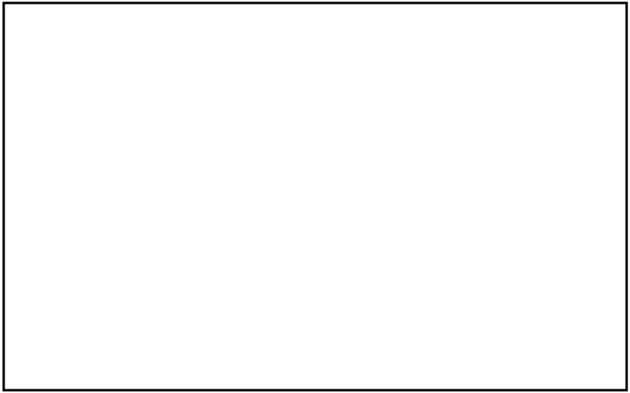
- The crucial point here is that even though we added together an **infinite** number of lengths, we ended up with a **finite** length:

$$100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + ... = 111\frac{1}{9}$$

- This might seem strange, but mathematicians have become very comfortable with the idea that an infinite sum can sometimes have a finite result

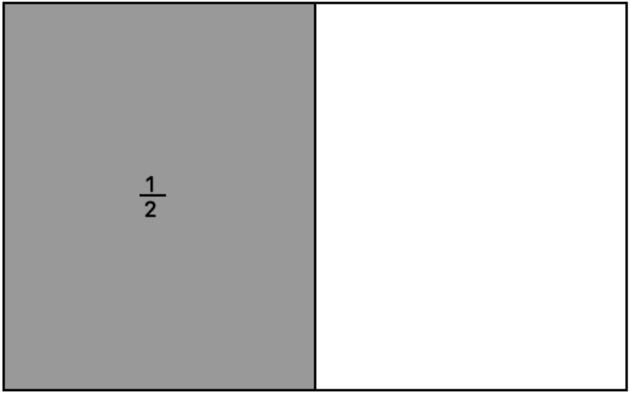
Infinite Sums

- $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ... = 1$



Infinite Sums

- $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots=1$



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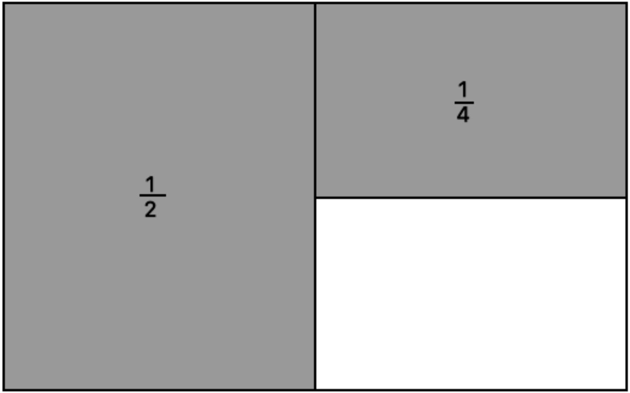

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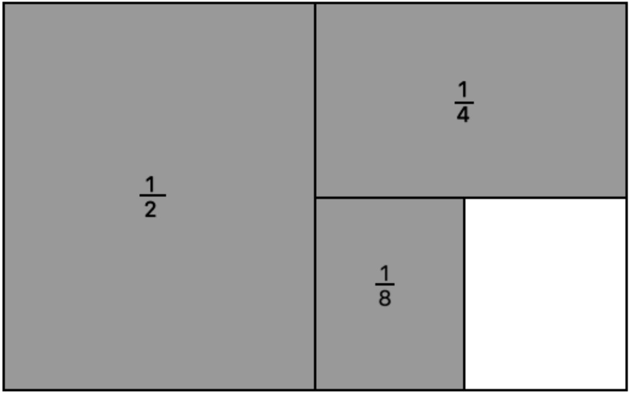

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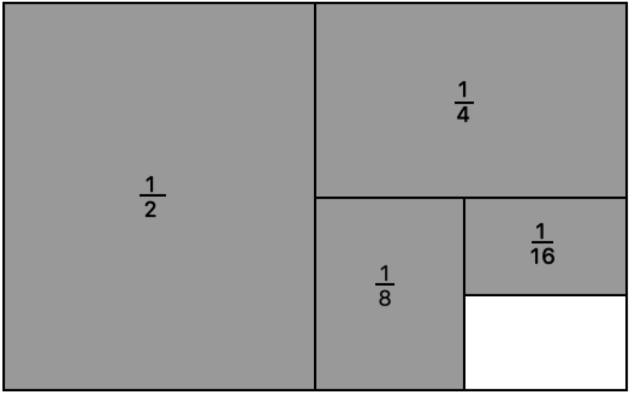

Infinite Sums

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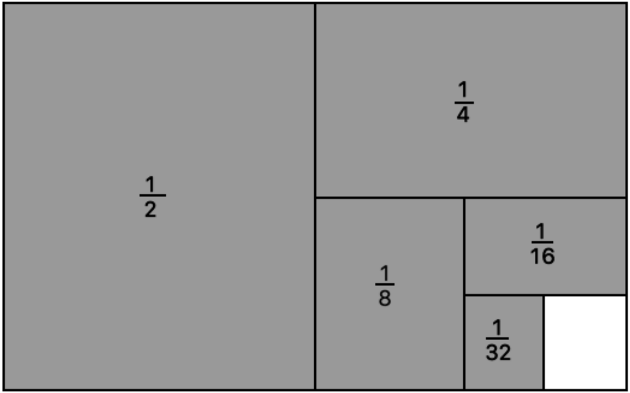

Quine's Diagnosis

the fallacy that emerges is the mistaken notion that any infinite succession of lengths has to add up to an infinite length. Actually when an infinite succession of lengths is so chosen that the succeeding lengths become shorter and shorter, the whole length may be either finite or infinite. It is a question of a convergent series.

Quine (1961) pp. 3–4 — modified to deal with lengths rather than times

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What is an Infinite Sum?

- The mathematical solution to the paradox of Achilles and the Tortoise points out that modern mathematicians are happy with the idea that an infinite sum can have a finite value
- But it is important not to misunderstand talk about "infinite sums"
- Mathematicians have not discovered that it is possible to add up infinitely many numbers **in exactly the same way** that we can add up two numbers
- Infinite sums are only **analogous** to finite sums

What is an Infinite Sum?

Let us be clear about what is meant by the assertion that the sum of the infinite series

$$100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \ldots$$

is $111\frac{1}{9}$. *It does not mean, as the naive might suppose, that mathematicians have succeeded in adding together an infinite number of terms. As Frege pointed out in a similar connection, this remarkable feat would require an infinite supply of paper, and infinite quantity of ink, and an infinite amount of time. If we had to add all the terms together, we could never prove that the series had a finite sum.*

What is an Infinite Sum?

> *To say that the sum of the series is* $111\frac{1}{9}$ *is to say that if enough terms of the series are taken, the difference between the sum of that* **finite number** *of terms and the number* $111\frac{1}{9}$ *becomes, and stays, as small as we please. (Or to put it another way: Let n be any number less than* $111\frac{1}{9}$*. We can always find a finite number of terms of the series whose sum will be less than* $111\frac{1}{9}$ *but greater than n.)*
>
> *Black (1950–1) p. 70*

Infinite Sums versus Finite Sums

- $100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + ... = 111\frac{1}{9}$
- This does not mean that when all these infinitely many numbers are stuck together, you get $111\frac{1}{9}$
- It just means that if you keep adding more and more numbers from this series together, you get closer and closer to $111\frac{1}{9}$
- Importantly, though, you never actually finish, and get to $111\frac{1}{9}$!

Back to Achilles

- But now it seems like Achilles faces a new, much harder problem
- In order to catch up to the tortoise, he must first move 100m, then 10m, then 1m, then $\frac{1}{10}$m...
- Thus in order to catch up with the tortoise, he must complete **infinitely** many tasks in a **finite** period of time!
- In the contemporary philosophical vernacular: Achilles must complete a **supertask**

The Racetrack

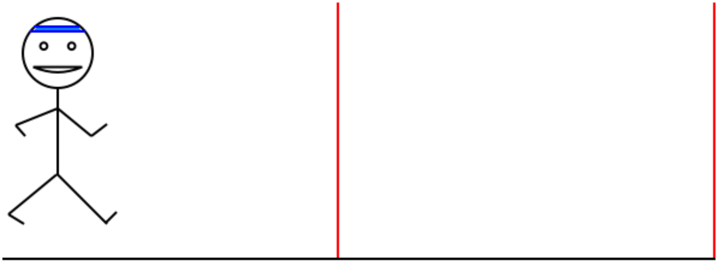
- Zeno illustrated this problem with the Racetrack Paradox

The Racetrack

- A runner wants to race to the finishing line

The Racetrack

- But first he must cross the midway point



The Racetrack

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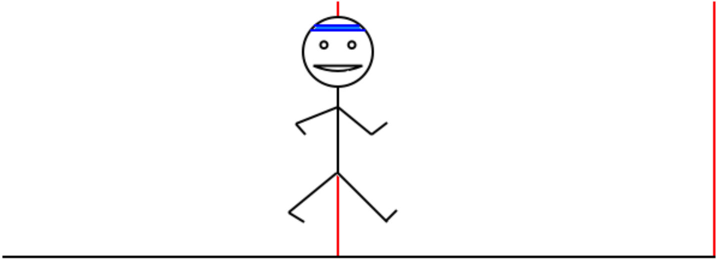
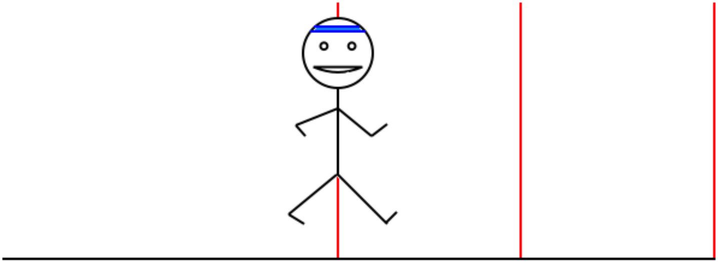


The Racetrack

- But now he needs to cross the new midway point



The Racetrack

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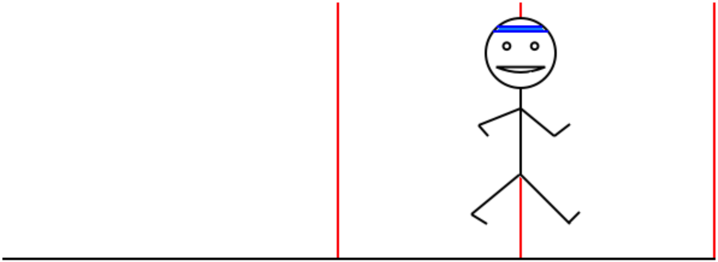
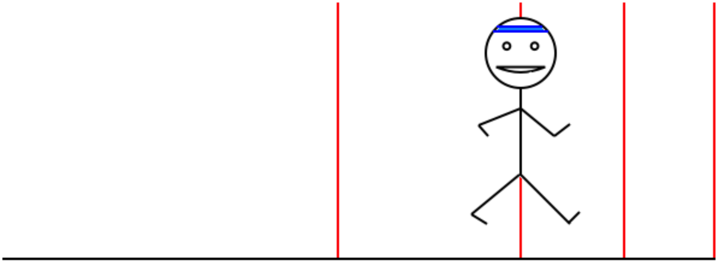


The Racetrack

- And then the new midpoint...



The Racetrack

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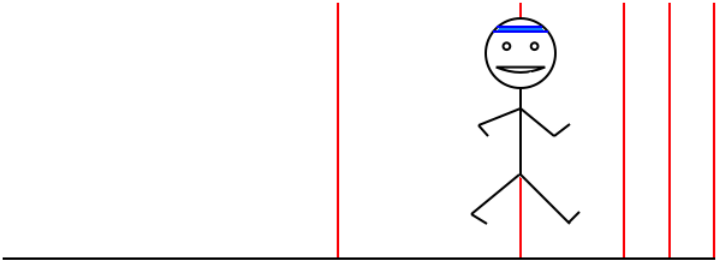

The Racetrack

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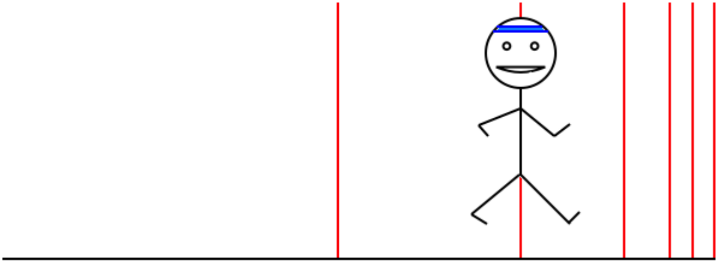
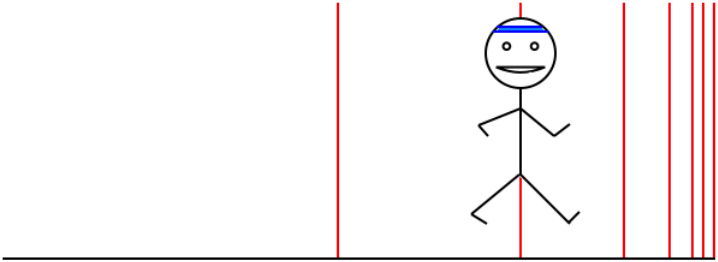


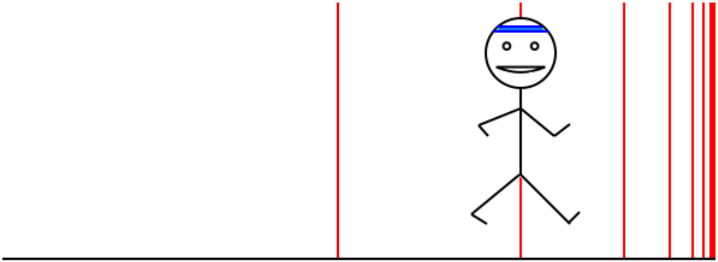
The Racetrack

- And then the new midpoint...



The Racetrack

- And infinitely many points between him and the finishing line

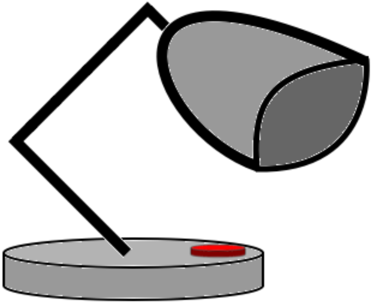


Are Supertasks Possible?

- Is it possible for the runner to do an infinite number of things (pass infinitely many points) in a finite period of time?
- It certainly sounds odd to say that he could, but is it actually incoherent?
- Thomson (1954–5) presented a famous thought experiment which seems to suggest that it is incoherent

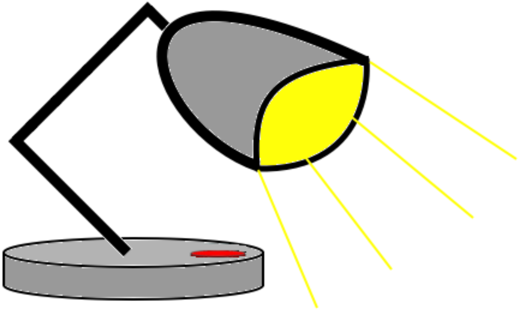
Thomson's Lamp

- Imagine a desk lamp



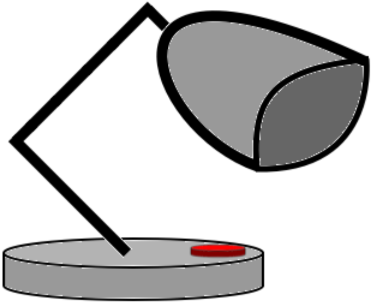
Thomson's Lamp

- You can turn it on by pushing its button



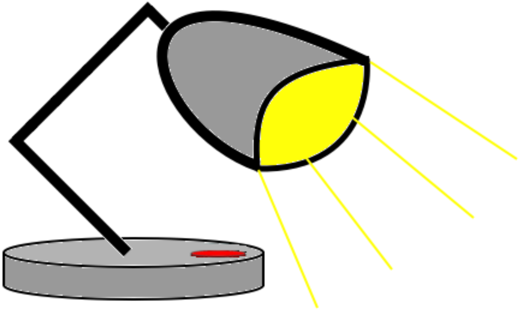
Thomson's Lamp

- And you can turn it off by pushing its button again



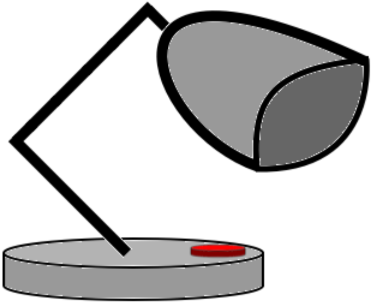
Thomson's Lamp

- Imagine you pushed the button once in a minute



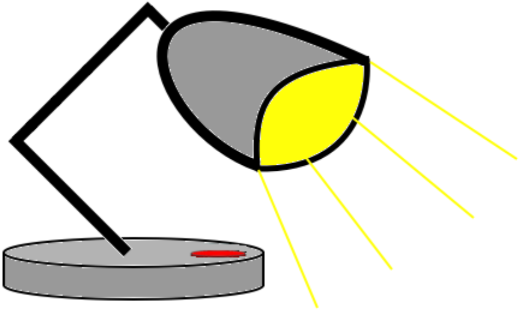
Thomson's Lamp

- And then again 30 seconds later



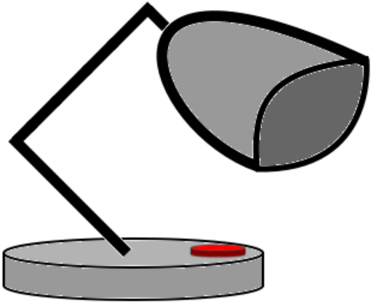
Thomson's Lamp

- And then again 15 seconds later



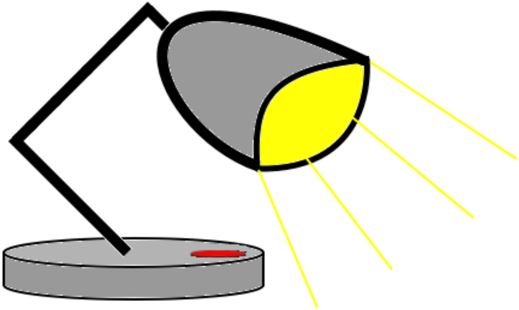
Thomson's Lamp

- And then again 7.5 seconds later



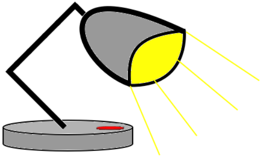
Thomson's Lamp

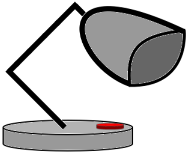
- And so on, taking half as long each time



Thomson's Lamp

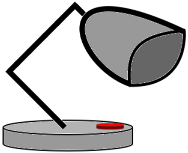
- After 2 minutes of this, will the lamp be on or off?



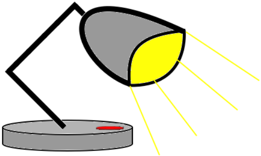


Thomson's Lamp

- It can't be on, because we never turned it on without turning it off again



Thomson's Lamp



- It can't be off, because we never turned it off without turning it on again

Thomson's Lamp

- We seem to be forced to say that Thomson's Lamp is **neither** on **nor** off at 2 minutes
- But that is (as good as) a contradiction: the lamp has to be on or off!
- This appears to show that believing in supertasks leads to contradictions

Incompleteness, not Inconsistency

- Benacerraf (1962) argues that the story of the lamp isn't really contradictory at all
- The story contains enough information to tell us whether the lamp was on or off at any time before the 2 minutes are up
  - Is the lamp on at 1.99 minutes?
  - I don't know, but we could work it out using the story!
- But the story does not tell us anything about what happens **once the 2 minutes are up**
- The story is not inconsistent, just incomplete
- We can consistently expand the story by adding either of
  - Once the 2 minutes were up, the lamp was on
  - Once the 2 minutes were up, the lamp was off

The Problem of Inexplicable Facts

- However, this does not show that there is nothing strange about the lamp story
- Suppose that the lamp is on after 2 minutes
- There is no way to explain **why** it is on
- If there were any way of explaining that, it would have to do with all the button pushing we were doing for those 2 minutes
- So it will just be a brute, inexplicable fact that the lamp ended up on

  (The same goes if we supposed the lamp was off)
- So even though this supertask does not lead to straightforward inconsistency, it does require inexplicable, brute facts

Zeno's Paradoxes

Zeno of Elea

Achilles and the Tortoise

The Racetrack

The Arrow

Is Space Infinitely Divisible?

- In order to get the Racetrack problem going, you need to assume that space is infinitely divisible
  - This does **not** mean that we have to assume that space is made up of infinitely small points
  - What it **does** mean is that we have to assume that we can go on dividing space into smaller and smaller parts forever
- Maybe that is our mistake?

Black on Infinite Divisibility

If it really were necessary for [Achilles] to perform an infinite number of tasks, as Aristotle says "to pass over or severally to come in contact with infinite things" (Physics, 233a), it would indeed by logically impossible for him to pass the tortoise. But the things he really does are finite in number; a finite number of steps, heart beats, deep breaths, cries of defiance, and so on. The track on which he runs has a finite number of pebbles, grains of earth, and blades of grass, each of which in turn has a finite, though enormous number of atoms. For all of these are things that have a beginning and end in space or time.

Black on Infinite Divisibility

> *But if anyone says we must imagine that the atoms themselves occupy space and so are divisible "in thought," he is no longer talking about spatio-temporal things. To divide a thing "in thought" is merely to halve the numerical interval which we have assigned to it. Or else it is to suppose what is in fact physically impossible beyond a certain point, the actual separation of the physical thing into discrete parts.*
>
> *Black (1950–1) pp. 79–80*

Is Space Infinitely Divisible?

- So maybe the problem is that we take our mathematical description of space (and time) too seriously
- The abstract, mathematical line that we use to represent spatial distances **is** infinitely divisible
- But maybe **real** space isn't
- Unfortunately, Zeno has a paradox for that assumption too!

The Arrow

- Suppose that space and time are made out of a series of consecutive, indivisible points
- Now consider an arrow in flight at a given instant

The Arrow

- The arrow cannot move during that instant
  - If it did, then it would be in different locations at the start and end of the instant
  - And in that case, the instant would have to have some duration that could be divided
- This point can be generalised: the arrow cannot move at any instant
- But we started with the supposition that time is just a series of indivisible instants
- So the arrow cannot move at all!

A Solution

- Motion is not something that happens **at** an instant
- Motion is a matter of being in **different** positions at **different** instants
- The arrow moves by virtue of being in different positions at different instants

When Does Motion Happen?

- We might feel a little bit uneasy about this solution
- Just **when** does motion happen?
- Not at any one instant
- Bergson (1911, p. 64) describes this as admitting '*a priori* the absurdity that movement coincides with immobility'

When Does Motion Happen?

- But this is not really absurd
- We just need to be careful not to equivocate
- In **one** sense the arrow is not moving at an instant, because motion takes time
- But in **another** sense, the arrow is moving at an instant, because it is in a different place than it was in the previous instant, and it will be in a different place again in the next instant
  - We might say that the arrow is moving **at** an instant, but not **in** an instant

Have we Solved Zeno's Paradoxes?

- Is the solution to Zeno's paradoxes to deny that time and space are infinitely divisible?
- Maybe that does solve the problems, but it is still a bit strange
- We would have ended up with a conclusion about the structure of space and time, but not by actually doing **experiments**
- We would have got there just by thinking really hard
- We might worry that this is not the right sort of job for philosophical reflection

This Week's Seminar

- Please read the following aritcles for this week's seminar:
  - Thomson, J (1954) 'Tasks and Super-Tasks', *Analysis*
  - Benacerraf, P (1962) 'Tasks, Super-Tasks and the Modern Eleatics', *Journal of Philosophy*
- Both of these articles are available via the Reading List on the VLE
- I have also posted some study questions to the VLE. Please bring short **written** answers to those questions

Next Week

- We will start looking at the Sorites Paradox
- Required Reading:
  - Sainsbury's *Paradoxes*, Chapter 3, §§3.1–3.3

References

- Benacerraf, P. (1962) 'Tasks, Super-Tasks and the Modern Eleatics', reprinted in Salmon (2001), pp. 103–29
- Bergson, H (1911) 'The Cinematographic View of Becoming', reprinted in Salmon (2001), pp. 46–66
- Black, M. (1950–1) 'Achilles and the Tortoise', reprinted in Salmon (2001), pp. 5–44
- Quine, WVO (1961) 'The ways of paradox', reprinted in Quine's *The Ways of Paradox and Other Essays* (1976), pp. 19–21
- Salmon, W (2001) *Zeno's Paradoxes*, 2nd edition (Indianapolis, IN: Hackett)
- Thomson, J. (1954–5) 'Tasks and Super-Tasks', reprinted in Salmon (2001), pp.89–102
- —— 'Comments on Professor Benacerraf's Paper', in Salmon (2001), pp. 130–8