Theories 4
The Frege-Hilbert correspondence

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Last time

- We looked at the axiomatisation of Euclidean geometry
- We then considered a non-Euclidean geometry, *hyperbolic geometry*
- After that we saw that hyperbolic geometry is consistent relative to the theory of real numbers
- Lastly, we saw a possible connection between Hilbert's claim that a consistent set of axioms defines the non-logical constants they contain and the completeness of a deductive system

Today's lecture

An overview of the correspondence

- Frege and Hilbert met at the 67th Convention of German Scientists and Doctors in September 1895
- Between 1/10/1895 and 7/11/1903, Frege and Hilbert exchanged a number of letters
- It starts innocuously enough, with Frege justifying the increasing use of formalisation in mathematics, logic and science. Hilbert heartily agrees
- But in his letter dated 27/12/1899, Frege criticises Hilbert's *Die Grundlagen der Geometrie*
- Hilbert defends himself, and Frege attempts to re-express his point
- Hilbert does not bother to respond to Frege's second attempt
- It is hard not to feel as though Hilbert became bored of Frege

Die Grundlagen der Geometrie: a reminder

- Hilbert axiomatised Euclidean and hyperbolic geometry
- He then showed that both of these theories were consistent relative to the theory of real numbers
- He does this by showing that if we re-interpret the non-logical primitives of these theories in terms of real numbers, then their axioms become theorems of real number theory
- If real number theory is consistent, then so must the axioms of Euclidean and hyperbolic geometry on these interpretations
- But consistency pays no attention to the meanings of the non-logical primitives, and so they must be consistent full stop.
- Hilbert also thinks of the axioms of Euclidean geometry as implicitly defining the non-logical primitives of Euclidean geometry, and the axioms of hyperbolic geometry implicitly defining the non-logical primitives of hyperbolic geometry

Frege on implicit definition: 27/12/1899

- Frege criticises both Hilbert's idea that axioms implicitly define non-logical primitives and Hilbert's relative consistency proofs
- One of Frege's objections to implicit definition can be put as a dilemma
- Either all the expressions in Hilbert's axioms have meanings, or at least some don't
  - If they do, then those axioms cannot serve to define a meaning for the non-logical primitives: they already have meanings!
  - If some don't, then the "axioms" cannot really be axioms: axioms can be true or false, but if Hilbert's "axioms" have meaningless parts then they cannot be true or false!
- Moreover, Frege says that Hilbert's "axioms" cannot be definitions because
  - Each non-logical primitive features in more than one of Hilbert's "axioms"
  - The axioms feature more than one non-logical primitive

Frege on Hilbert's relative consistency proofs: 27/12/1899

- First, Frege says that there is no need to prove the consistency of a set of axioms
- Following the old use of "axioms", Frege thinks that axioms need to be true. But if they are all true, then they must be consistent! (Truth implies consistency)
- Frege then says that Hilbert's proof that the axioms of Euclidean geometry are independent of each other (i.e. do not entail each other) involves adopting a higher stand-point from which Euclidean geometry is a special case of a more general theory
- By that he means that we see Euclidean geometry, which is about lines and points and the like, as a special case of a theory which can also be about real numbers, as on Hilbert's re-interpretations
- But, if the axioms of Euclidean geometry really are axioms, then their meanings must be fixed, and so we can't take on this higher stand-point

Hilbert on implicit definition: 29/12/1899

- Hilbert responds to Frege's objection about implicit definition by claiming that

  every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points I think of some system of things, e.g. the system: love, law, chimney-sweep... and then assume all my axioms as relations between these things, then my propositions, e.g. Pythagoras' theorem, are also valid for these things. In other words: any theory can always be applied to infinitely many systems of basic elements

Hilbert on implicit definition: 29/12/1899

- The idea is that the non-logical primitives in any theory be thought of as schematic. The theory just tells you what logical relations must hold between those non-logical primitives
- When we want to apply these abstract theories to a particular system, we interpret the non-logical primitives to stand for elements of that system

Hilbert on consistency and truth: 29/12/1899

- Perhaps the most striking thing in this letter from Hilbert is that he reverses what Frege says about consistency
- Frege said that the truth of a set of axioms guarantees their consistency
- Hilbert replied

  If the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence

- To repeat, last week we saw that today we might connect this thought with the fact that in a complete deductive system, every consistent set of sentences has a model (i.e. some interpretation makes all of the sentences in the set true)

Frege on implicit definition: 06/01/1900

- Frege is still concerned that more than one non-logical primitive features in each axiom of Hilbert's axiom-cum-definitions
- Frege says

  Your system of definitions is like a system of equations with several unknowns, where there remains a doubt whether the equations are soluble and, especially, whether the quantities are uniquely determined

- Some equations, like '$x + y + z = 85$' and '$x \times y = 49$', do not uniquely determine a meaning for 'x', 'y' and 'z'. In this example, it could be that $x = 7$, $y = 7$ and $z = 71$, or it could be that $x = 49$, $y = 1$ and $z = 35$
- Frege is saying that for all we know, Hilbert's definitions may be like this. We have no reason to think that they settle one meaning for each of non-logical primitives

Frege on implicit definition: 06/01/1900

- Frege goes on to say that Hilbert's axioms do not really define the first-level concepts *point*, *line* etc, but second-level concepts
- For Frege, a first-level concept is a property of objects. A second-level concept is a property of first-level concepts, and so on
- Frege's claim is that rather than defining properties of objects, Hilbert's axioms define relations which hold between the first-level concepts *point*, *line* etc

Frege on Hilbert's relative consistency proofs: 06/01/1900

- Frege says that for Hilbert, it is wrong to talk about Euclidean geometry as if it is just one thing
- It is a different theory on every different interpretation: when it is about points and lines it is one theory, T^1, when it is about real numbers it is another, T^2. Only the wording remains the same
- Hilbert proves the mutual independence of the axioms of T^2. But, Frege argues, we cannot infer from that that the axioms of T^1 are mutually independent

Frege on consistency and truth: 06/01/1900

- Frege absolutely refuses to accept that the consistency of some axioms implies their truth
- But he goes on to say that even if that were true, it would be useless
- This is because he claims that the only way to prove that some properties do not contradict one another is to present an object which satisfies them all
- So even if the consistency of some axioms entails its truth, I cannot establish the consistency of those axioms until I find some things which satisfy them

What is going on!?

- The correspondence is, then, a bit of a mess
- We need to figure out what is going on
- I will largely rely on Patricia Blanchette's 1996 paper 'Frege and Hilbert on consistency' in *The Journal of Philosophy*

Today's lecture

Syntactic consistency

- Hilbert is the easier character of the two for us to understand today
- This is because his conception of logic has become orthodoxy
- For Hilbert, logical relations hold between *sentences*, i.e. strings of symbols
- Whether a sentence ψ is entailed sentence ϕ is not a matter of what they mean
- Entailment is characterised syntactically
- As you'd expect, Hilbert has a syntactic notion of consistency
- A set of sentences Γ is consistent iff there is no ϕ such that Γ syntactically entails ϕ and $\neg\phi$

Properties and axioms

- There is a related notion of property-consistency
- Take the following two sets of axioms

(Σ1) $\{\forall x \exists y Rxy; \neg \exists y \forall x Rxy\}$
(Σ2) $\{\neg \forall x \exists y Rxy; \exists y \forall x Rxy\}$

- We can interpret '*Rxy*' as standing for many different relations
- And you all learnt in your 1A logic classes that many interpretations of '*Rxy*' make both members of (Σ1) true, e.g. $x = y$, but none make both members of (Σ2) true
- We can think of (Σ1) and (Σ2) as defining complex properties of a relation R
- If there are n non-logical primitives in a set of sentences, then that set will define an n-place complex property
- Recall Hilbert's idea that axiom sets provides a scaffolding of concepts; the property defined by the axiom set is that scaffolding

Property-consistency

- We say that a complex property defined in this way by some axioms is consistent iff some series of concepts/relations could satisfy it
- The property defined by ($\Sigma 1$) is consistent, whereas the one defined by ($\Sigma 2$) is not
- Call any sequence of concepts/relations which satisfy a property defined by a set of axioms Σ a Σ-structure
- So, the relation $x = y$ is a ($\Sigma 1$)-structure
- Hilbert's syntactic consistency proofs obviously establish property-consistency
- And so long as the deductive system is complete, property-consistency entails syntactic consistency

Today's lecture

Language

- Frege had a famous three-fold division
- First, there is the level of language
- This features sentences, thought of as strings of symbols
- It also has names, like 'Frege' and 'Hilbert'
- There are also first-level predicates, which are generated by replacing some of the names in a sentence with a variable. So from 'Frege is a logician' we can get the first-level predicate 'x is a logician'
- There are also second-level predicates, which are generated by replacing some of the first-level predicates in a sentence with variables. So from '$\forall x$(x is a logician)' we can generate the second-level predicate '$\forall x(Fx)$'
- Then there are third-level predicates, and so on

Reference

- There is also a level of reference, which is the level of the world
- Names refer to objects. So ‘Frege’ refers to an object, namely Frege
- First-level predicates refer to first-level concepts, second-level predicates to second-level concepts, etc. (Don’t be misled by the word ‘concept’!)
- Frege also thought of sentences as referring to truth-values, but don’t worry too much about that

Sense

- But in between language and the world, there were what Frege called *senses*
- It is hard to say exactly what senses are (it may even be impossible). But roughly, the sense of an expression is meant to be the way that that expression presents its referent
- So, the sense of 'Frege' is the way that that name presents its referent, Frege. 'Frege' and 'the author of *Die Grundlagen der Arithmetik*' both refer to the same thing, but have different senses, i.e. they present their referents differently
- Frege called the senses of whole sentences *thoughts*, but we would today more naturally call them propositions

Propositions and logic

- For Frege, logical relations don't really (or primarily) hold between sentences, but between propositions
- So his notion of logical consequence (Frege-consequence) is not syntactic
- But it is not what we would now call semantic: that is essentially connected to property-consistency. Our modern day notion of semantic consequence is at the level of reference
- Instead, Frege's notion of logical entailment was at the level of sense
- Equally, consistency for Frege is at the level of sense

Sentences and propositions

- Of course, Frege didn't ignore sentences: he invented quantified logic!
- For Frege, a good formal deductive system is such that for any sentences ϕ and ψ, ψ is derivable from ϕ iff the proposition expressed by ψ is a Frege-consequence of the proposition expressed by ϕ
- But, for Frege it is important that syntactic derivability and Frege-consequence can come apart in a particular way
- This because although every sentence expresses exactly one proposition, propositions can generally be expressed by many sentences
- If one proposition can be expressed by sentences with different degrees of syntactic complexity, then these two sentences will enter into different syntactic relations with other sentences

Sentences and propositions

- For example, the two sentences ‘Frege is a bachelor’ and ‘Frege is a man and has never been married’ both express the same proposition
- ‘Frege is a man’ is syntactically derivable from ‘Frege is a man and has never been married’, and so the proposition expressed by the former is a Frege-consequence of the proposition expressed by the latter
- But the sentence ‘Frege is a man’ is not *syntactically derivable* from ‘Frege is a bachelor’, even though we have already seen that the proposition expressed by the former is a Frege-consequence of the proposition expressed by the latter
- So, the fact that ψ is not syntactically derivable from ϕ does not show that the proposition expressed by ψ is not a Frege-consequence of the proposition expressed by ϕ

Today's lecture

Different conceptions of logic

- We have seen that Frege and Hilbert have different conceptions of logic
- It isn't surprising, then, that they found it so hard to understand each other
- Hilbert's achievements are undeniable, and easy for us to understand today
- So all I will try to do now is explain why Frege was so dissatisfied

Frege on relative consistency I

- When Frege asks whether the axioms of Euclidean geometry are independent of one another, he is asking a question about propositions
- He wants to know whether the propositions expressed by the axioms are Frege-consequences of each other
- Hilbert shows that the axioms of Euclidean geometry are syntactically independent of each other by interpreting them as being about real numbers
- But on these re-interpretations, the axioms express propositions about real numbers, not about points, lines etc.
- So for Frege, Hilbert's strategy involves changing the subject

Frege on relative consistency II

- But not only that, Frege has reason to be sceptical of any kind of consistency proof
- For Frege, to show that a set of propositions Π is consistent is to show that there is no proposition p such that p and $\neg p$ are Frege-consequences of Π
- But we cannot do that by finding a set of sentences Γ which expresses the propositions in Π and showing that there is no sentence ϕ such that ϕ and $\neg\phi$ are both syntactically derivable from Γ
- As we saw, it is not in general the case that if ϕ is not syntactically derivable from a set of sentences Γ then the proposition expressed by ϕ is not a Frege-consequence of the set of propositions expressed by the sentences in Γ
- Recall that {Frege is a bachelor; Frege is not a man} is syntactically consistent, but the proposition that Frege is a man and the proposition that Frege is not a man are both Frege-consequences of the propositions expressed by that set

Frege on implicit definition

- Frege insists that Hilbert's axioms do not define a meaning for the non-logical primitives they contain, i.e. 'x is a point', 'x is a line', etc.
- But he accepts that they together define a second-level concept
- This is because if we see all of the non-logical primitives in Hilbert's axioms as variables, then we see the whole "axioms" as second-level predicates
- A conjunction of second-level predicates stand for a second-level concept
- Frege, then, thinks that Hilbert is right that the axioms define something, but that he is confused about what they define
- Hilbert is confused because he didn't pay a close enough attention to the difference between first and second-level concepts

Frege on consistency and truth

- Frege insists that the only way to show that some concepts/relations are mutually consistent is by finding something which satisfies those concepts
- This gels nicely with his idea that axioms define second-level concepts
- The way that Hilbert shows that the second-level relation defined by the axioms of hyperbolic geometry, H, is consistent is precisely by finding some first-level concepts/relations which satisfy H: in particular, he finds some concepts/relations of real numbers which satisfy H
- In other words, when we find a model of a set of (first-order) axioms, we show that some structure of concepts/relations satisfy the second-level relation defined by those axioms

Who wins?

- In a sense, Hilbert won the debate with Frege
- His methods were undeniably fruitful, and his conception of logic became standard
- So standard, that now it is hard for us to even understand what Frege meant
- But Frege did have some valuable points to make
- Hilbert seems just wrong to think that his axioms defined the non-logical primitives they contained: they defined second-level relations
- And there is a good question to ask: should we prefer Hilbert's notion of logic, Frege's, or should we use both?