Theories 3
Hilbert and geometry

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Last time

- We gave a formal definition of axioms and axiom sets
- This involved giving an account of “finite specifiability”
- We also noted that axioms need not be true!
- We then looked at the axioms of Peano Arithmetic

Today's lecture

Euclidean geometry

Hyperbolic geometry

Relative consistency

Implicit definition

Geometry

- Geometry is the theory of points, lines, planes and the relations between them
- Geometry predates the ancient Greeks: the ancient Egyptians, Babylonians and Chinese all had geometrical knowledge
- But before the Greeks, geometry was a collection of rule-of-thumb procedures whose adequacy was to be assessed empirically
- It was the Greeks who first set out to deductively derive geometry from a set of axioms
- Euclid was not the first to contribute to this project. He was, however, the most successful
- In his masterpiece, *Elements*, Euclid attempted to derive 465 theorems from five axioms

Common notions

- In fact, it is not quite right to say that Euclid attempted to derive all of his theorems from the five axioms alone. He also made use of twenty-three definitions and five of what he calls ‘common notions’
- We might think that the common notions were intended to be what we now think of as logical axioms
- But as we have seen, all logical axioms are automatically members of every theory: any set of sentences entails every logical axiom
- So, if it is right to think of the common notions as logical axioms, then there is no need to explicitly add them

Definitions

- Euclid's definitions seem to be of (at least) two sorts
- Take Euclid's definitions of a point and of a line:
  - A point is that of which there is no part
  - A line is a length without breadth
- Now take his definition of a circle
  - A circle is a plane figure (one bounded by lines) which is bounded by a single line (a circumference) such that all of the lines radiating from the central point to the circumference are equal
- These two kinds of definition play different roles
- The first are meant to convey intuitive meanings, but they do no work within the formal system. They can't: as far as geometry is concerned, points and lines are the bedrock
- The second are meant to be usable within a system. For instance, from the definition of 'circle' we know which plane figures constructible within Euclid's geometry count as circles

Euclid's first four axioms

1. Given any two points P and Q, exactly one line can be drawn which passes through P and Q
2. Any line segment can be indefinitely extended
3. A circle can be drawn with any centre and any radius
4. All right angles are congruent to each other

Some comments on the axioms

- These axioms are, obviously, not written formally
- Moreover, they are written as permissions on what shapes can be drawn. We can rewrite them so that they are not of that form, e.g.

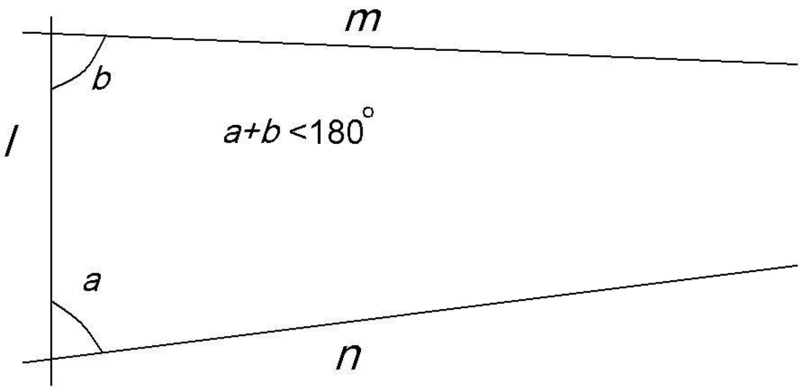
  $1'$ Between any two points there is a line

  (It can just make things a bit complicated!)
- The only non-logical primitives are:
  - point
  - line
  - lie on (as in 'two points lie on a unique line')
  - between (as in 'point C is between points A and B')
  - congruent
- Every non-logical expression in the axioms can be defined in terms of these

Axiom number 5

- Euclid's first four axioms always looked unobjectionable
- But now we come to axiom 5:

5 If a line l intersects two distinct lines m and n such that the sum of the interior angles a and b is less than 180° then m and n will intersect at some point


m
b
l
a+b <180°
a
n

The inadequacy of Euclid's axioms

- In fact, not all of Euclid's derivations followed from his five axioms
- In his proofs, Euclid makes frequent appeals to diagrams
- Some of his diagrams make it *really* look like a putative theorem followed from his axioms, but in fact it didn't
- Euclid's axiomatisation can, however, be repaired
- In one of his many masterpieces *Die Grundlagen der Geometrie*, David Hilbert properly axiomatised Euclidean geometry. He brought the total number of axioms up to twenty, along with nine non-logical primitives

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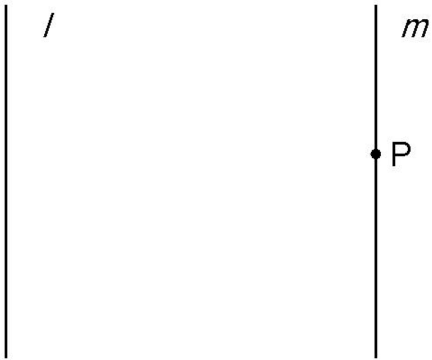
Playfair's axiom

- Axiom 5 was always looked on with suspicion. One reason was its complexity: it doesn't look as simple as the other axioms
- It turns out, though, that there is a more intelligible version of axiom 5, presented by Playfair:

5′ For every line *l* and for every point P that does not line on *l*, there is exactly one line *m* that can be drawn through P that is parallel to *l*

- Here '*x* is parallel to *y*' means '*x* and *y* at no point intersect'


l
m
P

But still...

- But still, plenty of people thought that axiom 5 looked dodgy
- Axioms 1–4 are, in a sense, abstractions from what we can do with a ruler, compass and protractor
- Axiom 5, however, does not look like such an abstraction
- Playfair's axiom tells us that if we have a given line l and draw two more lines m and n which intersect each other at some point, then at least one of m and n will intersect *somewhere* with l
- But we might have to go a *very* long way down the line to find this point of intersect
- In real life, we never really draw lines but line segments. Axiom 5 is not true of the "lines" we draw!

Deriving axiom 5

- As a result of the oddity of axiom 5, many mathematicians have attempted to derive it from axioms 1–4
- Eventually, though, a number of mathematicians started to explore the possibility that axiom 5 was not entailed by axioms 1–4
- They started working on a geometry which accepted all of the Euclidean axioms except for axiom 5; instead, they took the negation of 5 as an axiom:

$\neg 5$ There exists a line l and point P not on l such that at least two distinct lines parallel to l pass through P

- This geometry is called *hyperbolic geometry*
- It is a non-Euclidean geometry in the sense that it is a geometry but not Euclid's

History of hyperbolic geometry

- János Bolyai published a treatise on hyperbolic geometry in 1831. It was as an appendix to a book by his father, Wolfang Bolyai, who had actually spent much of his life trying to derive axiom 5 from 1–4
- Wolfang Bolyai was so proud of his son's work that he sent it to the greatest mathematician of the day, Carl Gauss, who was also Wolfang's friend
- Gauss didn't react as anyone expected: he claimed that he had already reached all of János's conclusions in unpublished work. We now know that he wasn't lying
- Nikolai Lobachevsky was actually the first to publish anything on hyperbolic geometry in 1829. But at first his work was little read: it was written in Russian and the few Russian mathematicians who read it criticised it fiercely
- In 1840, Lobachevsky's work was published in German, and was highly praised by Gauss

Some oddities of hyperbolic geometry

- For every line l and every point P not on l there are at least two distinct lines parallel to l which pass through P
- For any triangle ABC, the sum of the interior angles of ABC is strictly less than 180°
- There are no rectangles
- All similar triangles are congruent (i.e. there are no triangles of the same shape but different sizes)

Is hyperbolic geometry consistent?

- We have just seen that to our Euclidean eyes, hyperbolic geometry is very odd
- But no contradiction has been found
- But that is, of course, a long way of showing that hyperbolic geometry is consistent
- How is that to be established?

Today's lecture

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Hyperbolic geometry

Relative consistency

Implicit definition

Consistency and independence

- The task, then, to show that hyperbolic geometry is consistent
- Call the deductive closure of all the axioms of Euclidean geometry other than axiom 5 *neutral geometry*
- Another way to put the task is showing that neutral geometry does not entail axiom 5
- In other words, axiom 5 is *independent* of neutral geometry
- A sentence ϕ is independent of a set of sentences Γ iff $\Gamma \nvdash \phi$

Relative consistency

- It is actually quite a daunting task to prove that a theory is consistent once and for all
- Instead, we frequently settle for proofs of *relative consistency*
- If we can prove

  If Θ_1 is consistent then Θ_2 is consistent

  then we say that Θ_2 is consistent relative to Θ_1

How to show relative consistency

- We show a theory Θ_2 is consistent relative to a theory Θ_1 in two steps
- First, we give an interpretation of the non-logical primitives of Θ_2 in the language of Θ_1
- Second we show that so interpreted, the sentences of Θ_2 are all theorems of Θ_1
- If Θ_1 is consistent it follows that Θ_2 is consistent when understood in this new way
- But (syntactic) consistency pays no attention to meaning, and so if Θ_2 is consistent on this understanding of its primitives, then it is consistent on every understanding of its primitives
- So, we thereby show that if Θ_1 is consistent then Θ_2 is consistent

Hilbert's relative consistency proof

- In his *Die Grundlagen der Geometrie*, Hilbert proved that hyperbolic geometry was consistent relative to the theory of real numbers
- So, Hilbert first interprets the primitives of hyperbolic geometry in terms of real number theory, for example:
  - 'x is a point' is assgined the set of pairs $\langle x, y \rangle$ of real numbers
  - 'x is a line' is assigned the set of ratios $[u : v : w]$ of real numbers
  - 'x lies on y' is assigned the set of pairs $\langle \langle x, y \rangle, [u : v : w] \rangle$ such that $ux + vy + w = 0$
- He then shows that so interpreted, the axioms of hyperbolic geometry are theorems of real number theory

Hilbert's relative consistency proof

- Take the axiom
  - I(2) For any two points there exists at most one line on which those points lie
- Hilbert's task is first to re-interpret this axiom to become:
  - I(2)′ For any pair of pairs of real numbers $\langle\langle a, b\rangle, \langle c, d\rangle\rangle$, there is at most one ratio of real numbers $[e : f : g]$ such that $ae + bf + g = 0$ and $ce + df + g = 0$
- Then prove that I(2)′ is a theorem of real number theory

Today's lecture

Two kinds of definition

- Hilbert thought of his axiomatisations of Euclidean and hyperbolic geometries as *implicitly defining* the meanings of the non-logical primitives of those theories
- Normal, uncontroversial definitions are *explicit*
- They work by stipulating that a new symbol will be an abbreviation of another, already understood, symbol
- For example, we explicitly define the predicate 'x is even' when we stipulate that it is to have the same meaning as 'There is some integer n such that $x = 2n$'
- So, in explicit definitions we *mention* the symbol being defined
- An implicit definition, on the other hand, is meant to give a meaning to an expression by *using* that expression
- Hilbert's idea was that a consistent set of axioms implicitly defines the non-logical primitives they contain

Implicit definition and completeness

- "If the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist"
- There is a way of hearing this sentence which makes it sound true if we are using a complete deductive system
- A deductive system is complete iff (for any ϕ and Γ, if $\Gamma \vDash \phi$ then $\Gamma \vdash \phi$)
- It is provable that if our deductive system is complete, any consistent set of sentences is satisfiable
- Γ is consistent iff there is no ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$
- Γ is satisfiable iff there is no ϕ such that $\Gamma \vDash \phi$ and $\Gamma \vDash \neg\phi$
- In other words, Γ is satisfiable iff some interpretation makes all of the sentences of Γ true; we call such an interpretation a *model* of Γ

Implicit definition and completeness

Assume that our deductive system is complete and a set of sentences Γ is consistent

Now suppose that Γ is unsatisfiable

In that case there is some ϕ such that $\Gamma \vDash \phi$ and $\Gamma \vDash \neg\phi$

By completeness, there is some ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$

So Γ is inconsistent

Contradiction!

Hence Γ is satisfiable

Next time on *Theories*...

- So, if an axiom set is consistent, it has a model
- But whether this is *really* enough to vindicate Hilbert's idea that a consistent set of axioms implicitly defines the non-logical primitives it contains is an issue we will turn to next week