Theories 2
Axioms

Rob Trueman
rt295@cam.ac.uk

Fitzwilliam College, Cambridge

28/01/13

Last time

- We defined a *theory* as a deductively closed set of sentences
- We defined a *formal theory* as a theory expressed in a formal system accompanied with a formal deductive system
- We clearly distinguished between the syntactic notion of deduction, $\vdash$, and the semantic notion of logical consequence, $\models$

Today's lecture

Theories are infinitely big

- Typically, theories have infinitely many members
- A theory is a deductively closed set of sentences, and on most deductive systems infinitely many sentences can be deduced from any given sentence
- For example, in PL we can deduce '$\neg\neg P$', '$\neg\neg\neg\neg P$', '$\neg\neg\neg\neg\neg\neg P$'... from '$P$'
- We can even deduce infinitely many sentences from the empty set
- Again in PL, '$P \vee \neg P$', '$P \vee \neg\neg\neg P$', '$P \vee \neg\neg\neg\neg\neg P$'... can all be deduced from the empty set
- As a result, theories are typically infinitely big and so we cannot give a theory just by listing every sentence that is a member of that theory

Axioms

- Instead, we usually present a theory by presenting a set of axioms (or axiom set)
- I will generally use 'Σ' for axiom sets and 'α' for individual axioms (i.e. the members of axiom sets)
- An *axiom set* Σ for a theory Θ is a finitely specifiable set of wffs from which every theorem of Θ can be deduced
- In other words, Θ is the deductive closure of its axiom set Σ
- For any ϕ, $\phi \in \Theta$ iff $\Sigma \vdash \phi$
- We say that a the axiom set for a theory *axiomatises* that theory, and we say that a theory is *axiomatisable* iff some axiom set axiomatises it

Finite specifiability: what and why

- In our definition of 'axiom set', we said that axiom sets have to be "finitely specifiable"
  - What does that mean?
  - Why does an axiom set have to be finitely specifiable?
- A set of wffs Γ is *finitely specifiable* iff we can lay down explicit rules for determining of any given wff in a finite period of time whether or not it is in Γ
- (That is *not* the same as saying that we can list all of the members of Γ in a finite period of time!)
- The reason that our axiom set must be finitely specifiable is that it gives us finite beings the means to grasp infinite theories
- If axiom sets did not have to be finitely specifiable, every theory would be trivially axiomatisable: every theory would axiomatise itself!

Two ways of finitely specifying a set

- One way I could finitely specify an axiom set Σ is just to list a finite number of axioms and say that a sentence is a member of Σ iff it appears on that list
- But this isn't the *only* way. We can also use what are called *schemata* (the singular of which is *schema*)
- A schema is a *shape* or *form* of a sentence. For example,
  (i) $\phi \vee \neg\phi$
  (ii) $\exists x(\phi \wedge Fx)$
- Schemata have *instances*, which are sentences of the same shape. For example, '$P \vee \neg P$' is an instance (in PL) of (i), and '$\exists x(Gm \wedge Fx)$' is an instance (in QL) of (ii)
- When finitely specifying an axiom set, we can also stipulate that every instance of a given schema is a member of the set

Two ways of finitely specifying a set

- Importantly, a given schema typically has infinitely many instances. For example, take the PL schema we had earlier
  (i) $\phi \vee \neg\phi$
- We can plug *any* PL sentence we like into ϕ. But there are infinitely many sentences in PL. So (i) has infinitely many instances
  - $P \vee \neg P$
  - $Q \vee \neg Q$
  - $(P \vee \neg P) \vee \neg(P \vee \neg P)$
  - ...
- So, when we finitely specify an axiom set using schemata, we typically get axiom sets with infinitely many wffs as members
- It is not the case that a set is finitely specifiable iff it has finitely many members!

Multiple axiomatisations

- So, an axiom set Σ for a theory Θ is a set of finitely specifiable wffs such that for any ϕ, $\phi \in \Theta$ iff $\Sigma \vdash \phi$
- Importantly, in general theories can be axiomatised in different ways
- That is, there can be a theory Θ and two finitely specifiable sets of sentences Σ_1 and Σ_2 such that
  - For any ϕ, $\phi \in \Theta$ iff $\Sigma_1 \vdash \phi$
  - For any ϕ, $\phi \in \Theta$ iff $\Sigma_2 \vdash \phi$
  - $\Sigma_1 \neq \Sigma_2$

Multiple axiomatisations

- Let Θ be the deductive closure of the set of the sentences
  (i) $\forall x Rxx$
  (ii) $\forall x \forall y (Rxy \equiv Ryx)$
  (iii) $\forall x \forall y \forall z ((Rxy \wedge Ryz) \supset Rxz)$
- Obviously, the set of sentences (i), (ii) and (iii) is an axiom set for Θ
- But so is the set of sentences
  (ii) $\forall x \forall y (Rxy \equiv Ryx)$
  (iii) $\forall x \forall y \forall z ((Rxy \wedge Ryz) \supset Rxz)$
  (iv) $\forall x \exists y Rxy$

Today's lecture

Must axioms be true?

- Often we think of the axioms of a theory as self-evident truths
- For example, Euclid famously tried to give an axiomatisation of geometry; he thought of himself as giving us a way of deriving all of geometry from self-evident truths
- In fact, some theorems of geometry were not derivable from Euclid's axioms. (This is not to be confused with the discovery of non-Euclidean geometries!) Hilbert repaired Euclid's axiomatisation

Not in our sense of 'axioms'!

- We will not require that axioms be self-evident, or even true at all
- All that matters is that Σ be finitely specifiable and $\Sigma \vdash \Theta$
- In fact, things are not *that* simple. Hilbert argued that any consistent set of axioms defined the meanings of the non-logical symbols in the axioms, and then the axioms are bound to be true. More on that in the fourth lecture!

Today's lecture

A formal definition of axioms

Axioms and truth

Peano Arithmetic

Two properties of theories

Axiomatising arithmetic

- Just as Euclid wanted to axiomatise geometry, in the nineteenth century mathematicians and philosophers wanted to axiomatise arithmetic
- Arithmetic is the theory of the natural numbers 0, 1, 2, 3, 4... and the operations of addition and multiplication
- One reason to be interested in the axiomatisation of arithmetic is *logicism*. Logicism is the thesis that the arithmetical truths are all logical truths. Frege wanted an axiomatisation of arithmetic to establish logicism
- In 1888 Dedekind proposed an axiomatisation of arithmetic, and in 1889 Peano published a more precise formulation of those axioms. The deductive closure of those axioms are today known as *Peano Arithmetic*, or PA

The language of PA

- The formal language of PA, $\mathcal{L}_{\text{PA}}$, is just like $\text{QL}^{=}$ except it only contains four non-logical constants:
  - The individual constant '0'
  - The *successor functor* 'Sx'
  - The *addition functor* '$x + y$'
  - The *multiplication functor* '$x \times y$'
- The deductive system of PA is that of $QL^{=}$

Functors

- You may be unfamiliar with *functors* (or functional symbols)
- Just like a predicate, a functor has variables, which we can either replace with an individual constant or bind with a quantifier
- Unlike a predicate, the result of replacing the variable in a functor with an individual constant is not a sentence, but a complex individual constant referring to an object
- For example, when we put the individual constants '3' and '5' into the functor '$x + y$' we get the complex constant '$3 + 5$', which (on the intended interpretation) stands for the number 8
- A functor stands for a function, which maps one or more objects to an object

The intended interpretation of $\mathcal{L}_{\text{PA}}$

- $\mathcal{L}_{\text{PA}}$ has an intended interpretation
- The domain is meant to be the set of natural numbers: $\{0, 1, 2, 3 ...\}$
- '0' is meant to refer to 0
- '$S0$' is meant to refer to 1, '$SS0$' to 2, '$SSS0$' to 3, and so on; generally, 'Sm' is meant to refer to the number after m
- '$m + n$' is meant to refer to the sum of m and n
- '$m \times n$' is meant to refer to the product of m and n

The first two axioms of PA

- The first two axioms of PA concern the structure of the number series
  (1) $\forall x(0 \neq Sx)$
  (2) $\forall x \forall y(Sx = Sy \supset x = y)$
- (1) tells us that 0 does not come after any number
- (2) tells us that no two numbers have the same successor. If x and y have the same successor, then $x = y$

The second two axioms of PA

- The second two axioms of PA concern addition
  (3) $\forall x(x + 0 = x)$
  (4) $\forall x \forall y(x + Sy = S(x + y))$
- This is a *recursive definition* of addition
- (3) tells me how to do the simplest addition, that is adding 0. For example, from (3) I can infer that $S0 + 0 = S0$
- Now imagine I want to know what $S0 + S0$ equals. Well, from (4) I know that
  - $S0 + S0 = S(S0 + 0)$

  I already know from (3) that
  - $S0 + 0 = S0$

  So I can infer that
  - $S0 + S0 = SS0$
- In this way I can work out more and more complicated additions. By following the process through, I can work out any addition of any complexity

The third two axioms of PA

- The third two axioms of PA concern multiplication
  (5) $\forall x(x \times 0 = 0)$
  (6) $\forall x \forall y(x \times Sy = (x \times y) + x)$
- Again, this is a recursive definition of multiplication
- (5) tells me how to do the simplest multiplication, that is multiplying by 0. For example, from (5) I can infer that $S0 \times 0 = 0$
- Now imagine I want to know what $S0 \times S0$ equals. Well, from (6) I know that
  - $S0 \times S0 = (S0 \times 0) + S0$

  I already know from (5) that
  - $S0 \times 0 = 0$

  So I can infer that
  - $S0 \times S0 = 0 + S0 = S0$
- In this way I can work out more and more complicated multiplications. By following the process through, I can work out any multiplication of any complexity

Induction

- So far we have given six axioms. If we stopped there, we wouldn't have PA but what is called *Robinson Arithmetic* (RA)
- To get from RA to PA we have to add infinitely many axioms. Fortunately, though, they are all instances of the following schema

  (I) $((\phi(0) \wedge \forall x(\phi(x) \supset \phi(Sx))) \supset \forall x \phi(x))$
- The Induction Schema in essence tells us that if 0 has a property and, in general, a number has that property only if its successor has it too, then the property holds of every number
- This makes sense on the intended model of PA. Suppose I convinced myself that 0 has property F, and that in general, if m has F then Sm has F. Then $S0$ (i.e. 1) must have F, and so $SS0$ (i.e. 2) must have F, and so on for all of the numbers

Peano Arithmetic

(1) $\forall x(0 \neq Sx)$

(2) $\forall x \forall y(Sx = Sy \supset x = y)$

(3) $\forall x(x + 0 = x)$

(4) $\forall x \forall y(x + Sy = S(x + y))$

(5) $\forall x(x \times 0 = 0)$

(6) $\forall x \forall y(x \times Sy = (x \times y) + x)$

(I) $((\phi(0) \wedge \forall x(\phi(x) \supset \phi(Sx))) \supset \forall x \phi(x))$

Today's lecture

A formal definition of axioms

Axioms and truth

Peano Arithmetic

Two properties of theories

Consistency

- One property which we obviously want of a theory is that it be *consistent*
- Roughly, a theory Θ is consistent if there is no ϕ such that Θ entails ϕ and $\neg\phi$
- We might mean something syntactic or semantic by 'consistent'
  (i) Θ is consistent iff there is no ϕ such that $\Theta \vdash \phi$ and $\Theta \vdash \neg\phi$
  (ii) Θ is consistent iff there is no ϕ such that $\Theta \vDash \phi$ and $\Theta \vDash \neg\phi$
- Typically I will mean (i) by 'consistent'. We will use 'satisfiable' for (ii)
- If this ever changes, as it will when discussing Frege, I will make this clear

Consistency and axioms

- If Σ is an axiom set for a theory Θ, then Θ is consistent iff Σ is
- Every sentence in Θ can be deduced from Σ. As Θ is deductively closed, it follows that if a contradiction can be deduced from Θ, then it can be deduced from Σ

Negation completeness

- Another property that we might want a theory to have is *negation completeness*
- Roughly, a theory Θ is negation complete iff for every ϕ, either Θ entails ϕ or Θ entails $\neg\phi$ (or both)
- Again, this could be understood syntactically or semantically
  - (i) Θ is negation complete iff for every ϕ, $\Theta \vdash \phi$ or $\Theta \vdash \neg\phi$
  - (ii) Θ is negation complete iff for every ϕ, $\Theta \vDash \phi$ or $\Theta \vDash \neg\phi$
- Again, I will mean (i) by 'negation complete'

Gödel's incompleteness theorems

- Amazingly though, Gödel proved that PA cannot be both consistent and negation complete
- He did this by constructing a sentence, G, such that if PA is consistent, then PA $\nvdash G$ and PA $\nvdash \neg G$
- This tricky sentence G can be described as saying, roughly, 'I am not provable in PA'
- But to learn the details, you have to do Part II mathematical logic