Theories 1
What is a theory?

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Course outline

1. What is a theory?
2. Axioms
3. Hilbert and geometry
4. The Frege-Hilbert correspondence

Today's lecture

The definition of a theory

Syntax

Semantics

Deducibility vs consequence

Some things that we might mean by ‘theory’...

- A big picture, in principle account — the theory of general relativity
- An explanation — psychology
- A hypothesis — the theory of evolution
- A conjecture — Golbach’s conjecture
- A guess — folk physics
- Wishful thinking

…but that we won't mean by 'theory'

- We won't quite mean any of these things by 'theory'
- We are interested in what logicians normally call 'theories'
- 'Theory' in this sense is very general indeed; many things count as logician's theories which we might not normally think of as theories
- Of course, if we are interested in, say, theories as explanations, we could start with logician's theories and then ask what we have to add to get to explanations

What are theories composed of?

- Our first question on the road to the logician's notion of theories is, What are theories composed of?
- Here are three obvious suggestions
  (1) Beliefs
  (2) Propositions
  (3) Sentences
- In many ways, option (3) is the least intuitive. Theories are the sorts of things we can believe and can be expressed in many different languages. But, strictly speaking, sentences cannot be believed or expressed in different languages
- More generally, we might think of theories as having a fixed content. But sentences only have a content relative to an interpretation

Theories are sets of sentences

- Nonetheless, logicians take theories to be sets of sentences
- This is partly because we have a better understanding of sentences than beliefs or propositions
- In particular, we have a better understanding of the *structure* of sentences. At least when it comes to formal languages, we know what it means to say that a sentence is the conjunction of two others, but what it means to say that a proposition is the conjunction of two others is still debatable
- Logicians have also managed to use the fact that sentences can be interpreted in a number of ways to their advantage. But we will talk more about this issue more when we come to Hilbert

Deductive closure

- But although theories are sets of sentences, not *every* set of sentences is a theory
- We say that a set of sentences is deductively closed iff every sentence that can be deduced from that set is itself a member of that set
- We will say that a set A is the deductive closure of a set of sentences B iff every sentence which can be deduced from B is a member of A, and every member of A can be deduced from B.
- (If A is the deductive closure of B then B is a subset of A, as every sentence can be trivially deduced from itself)

The definition of a theory

- A *theory* is a deductively closed set of sentences
- We call the sentences in a theory *theorems*

An example

- So far we have been talking wholly in the abstract, so let's give a concrete example
- It is important to note just how general this definition of a theory is. *Any* deductively closed set of sentences is a theory
- So let's take as our example the deductive closure of the set of the following two sentences:
  (1) There has never been a good Spiderman film
  (2) At least some Batman films were good
- Call this theory A. Every sentence that is deducible from (1) and (2) together are members of A, and *vice versa*
- So, A also contains the sentence 'There has never been a good Spiderman film and at least some Batman films were good', 'It is not the case that it is not the case that there has never been a good Spiderman film', and so on

Today's lecture

Formal theories

- Theories are deductively closed sets of sentences. But natural language sentences are messy and hard to put to logical work
- So, logicians are mostly concerned with formal theories
- A theory is *formal* iff all of its theorems are in a formal language with a formal system of deducibility

Formal languages

- A formal language is a language of which we can give a precise syntactic description
- The *syntax* of a language is a characterisation of that language as a system of signs. It pays absolutely no attention to what those signs might mean, or to whether they are true or false, etc. etc.

Vocabulary

- A syntax for a language supplies a *basic vocabulary* and *formation rules*
- The basic vocabulary are the symbols which we do not break down any further
- The basic vocabulary of PL consists of:
  - Atomic sentences letters, e.g. 'P' and 'Q'
  - Logical operators, e.g. '$\neg$' and '$\wedge$'
- The basic vocabulary of QL consists of:
  - Atomic sentences letters
  - Atomic predicates, e.g. 'Fx' and 'Rxy'
  - Names, e.g. 'm' and 'n'
  - Variables, e.g. 'x' and 'y'
  - Logical operators, which now include the quantifiers '$\exists$' and '$\forall$'

Formation rules

- The formation rules tell us how to build well-formed formulae (or wff) from well-formed formulae
- For example, in PL it is a formation rule that if ϕ and ψ are wffs then so is $\phi \wedge \psi$. In QL, it is a formation rule that if ϕ is a wff and α is a variable, then $\forall \alpha \phi$ is a wff
- The formation rules can be thought of as like the grammar of the language

Deduction

- As well as being expressed in a formal language, a formal theory has a formal, syntactic system of deduction
- We will call the *syntactic* logical entailment 'deduction'. To be clear, this is also what I meant in the definition of a theory by 'deduction'. But be warned: people sometimes mean something semantic by 'deduction'!
- A deductive system (or proof calculus) is a system of rules to determine which sentences can be deduced from other(s)
- These rules are only concerned with the signs, and not their meanings. The rules are just rules for the manipulation of signs
- For example, the deductive system for PL includes the rules for manipulating logical constants in a tree and the 'master' rule that a closed tree shows deducibility
- We can make up any rules we like, but as we will discuss, some rules are better than others!

Symbolism

- Typically, the sign 'Θ' will stand for theories, '$\mathcal{L}$' for formal languages, 'Δ' for deductive systems, 'ϕ' and 'ψ' for wffs, and 'Γ' for sets of wffs
- The relation '$\vdash$' (the single turnstile) stands for deducibility. Remember, deducibility is relative to deductive systems: one sentence might be deducible from another in some systems and not in some other systems
- '$\phi \vdash \psi$' means that ψ can be deduced from ϕ
- '$\Gamma \vdash \phi$' means that the formula ϕ can be deduced from the formulae in Γ
- '$\vdash \phi$' means that ϕ can be deduced from nothing, i.e. ϕ is a logical truth of Δ
- '$\Theta \vdash \phi$' means that ϕ is a theorem of Θ

A formal definition of formal theories

- Θ is a formal theory iff
  - $\Theta \subseteq \mathcal{L}$ (i.e. every sentence in Θ is a sentence of a formal language $\mathcal{L}$)
  - $\phi \in \Theta$ iff $\Theta \vdash \phi$ (i.e. Θ is deductively closed)

Today's lecture

Truth

- So far we have been concerned with the syntax of language, which just describes the language as a system of signs
- This is to be contrasted with the *semantics* of a language
- The semantics of a language fixes the truth-value of every sentence in the language
- When the sentences are complex it normally does this by assigning *semantic values* to the parts of the sentences
- Semantics is only concerned with truth-value and properties needed to fix truth-values

Interpretations

- A semantics for a language gives us a way to fix the truth-values of every sentence in the language
- We start with an *interpretation*
- An interpretation supplies a domain of quantification for the variables in the language (if any). The domain is a set
- It also supplies an assignment values to the primitives in our language
- For example, if our language is QL, the interpretation supplies a domain for the variables, D, and an assignment v which assigns objects in the domain to names, sets to predicates, and truth-values to atomic sentence letters. We identify the interpretation with the ordered pair $\langle D, v \rangle$

Intended interpretations

- Importantly, for any given formal language there are *lots and lots* of different interpretations
- An interpretation of QL, for example, can assign any set as the domain, any object in that domain as the referent of any name, and so on
- Sometimes we think of theories as having an *intended interpretation*. For example, we intend arithmetical theories to be interpreted as being about numbers
- However, it can be tricky to explain just what an intended interpretation is or how we grasp which interpretation is intended

Truth-values

- Our semantics then gives us a way of assigning truth-values to every sentence in the language on the basis of our interpretation. For example, the semantics for QL tell us that:
  - '$P \wedge Q$' is true iff 'P' is true and 'Q' is true
  - 'Fa' is true iff $v(\text{`}a\text{'}) \in v(\text{`}F\text{'})$
  - '$\forall x Fx$' is true iff every member of the domain is a member of $v(\text{`}F\text{'})$

Truth is relative to an interpretation

- Strictly speaking, truth is relative to an interpretation: on different interpretations, different sentences are true
- This is not a scary relativity. It is just the common sense truth that a sentence may be true on one way of understanding it and false on another
- (We *would* be left with a scary relativity if we said that there is no way to choose between the ways to interpret a given sentence in a given context)
- When people say that a sentence is true without explicitly mentioning an interpretation, they normally mean true relative to a salient interpretation; often, this is the intended interpretation

Logical consequence

- Along with our previous syntactic characterisation of logical entailment, we can give a semantic one
- We will call this semantic logical entailment ‘logical consequence’. But be warned: people sometimes mean something syntactic by ‘logical consequence’!
- ψ is a logical consequence of ϕ iff every interpretation that makes ϕ true makes ψ true

Symbolism

- Typically I will use '$\mathcal{I}$' to stand for interpretations
- The relation '$\vDash$' (the double turnstile) stands for logical consequence. It can be used to relate interpretations to formulae and formulae to formulae
- $\mathcal{I} \vDash \phi$ means that ϕ is true under interpretation *I*. In other words: $\mathcal{I}$ *satisfies* ϕ
- '$\mathcal{I} \vDash \Gamma$' means that $\mathcal{I}$ satisfies every member of Γ
- '$\phi \vDash \psi$' means that every interpretation that satisfies ϕ satisfies ψ
- '$\Gamma \vDash \phi$' means that every interpretation which satisfies every member of Γ satisfies ϕ
- '$\vDash \phi$' means that every interpretation satisfies ϕ

Today's lecture

The definition of a theory

Syntax

Semantics

Deducibility vs consequence

Keep deducibility and consequence distinct!

- You must keep the distinction between deducibility and logical consequence absolutely clear in your minds
- Deducibility is just concerned with the manipulation of signs, and no more
- Logical consequence is concerned with *truth* and *satisfaction*

Soundness and completeness

- With this distinction in mind, we can ask how deducibility and consequence relate to each other
- A deductive system is *sound* iff (if $\Gamma \vdash \phi$ then $\Gamma \vDash \phi$)
- A deductive system is *complete* iff (if $\Gamma \vDash \phi$ then $\Gamma \vdash \phi$)
- Any good deductive system will be sound; we just have to choose our deductive rules well. (However, it is tricky to prove that a system is sound!)
- But not every deductive system is complete. First-order logic, i.e. QL, is complete. As it is also sound, deducibility and consequence are materially equivalent
- On the other hand, full second-order logic, which contains sentences like '$\forall F\ Fa$', is not complete