

Intermediate Logic

Lecture Seven

Natural Deduction for FOL

Rob Trueman
rob.trueman@york.ac.uk

University of York

Natural Deduction for FOL

Introducing Natural Deduction for FOL

Universal Elimination

Existential Introduction

Universal Introduction

Existential Elimination

Reasoning about *all* Interpretations

- Reasoning about interpretations are great when we want to show that some argument is **not** valid in FOL, or that some sentence is **not** a logical truth
 - All we have to do is come up with a **single** counter-interpretation
- But when we want to show that an argument **is** valid in FOL, or a sentence **is** a logical truth, then they are a lot less helpful
 - To show that \mathcal{A} is a logical truth, we must somehow show that it is true in **all** interpretations

An Easy Example

- Exercise: show that $\models \forall x x = x$
- Consider an arbitrary interpretation, call it *INT*
- Let c be a new name added to the language
- On any way of extending *INT* to cover c , $c = c$ will come out true
 - No matter how we interpret c , it will always refer to the same object as itself!
- So by the semantic rules for universal quantification, ' $\forall x x = x$ ' will be true in *INT*
- But *INT* was an **arbitrary** interpretation, so ' $\forall x x = x$ ' will be true in every interpretation

Most Examples aren't that Easy!!!

- It is sometimes possible, then, to reason about **all** interpretations
- But most of the time, this is very, very hard
- There certainly is not any mechanical method for searching through interpretations (whereas there was a mechanical method for searching through TFL valuations)
- As a result, it is not very practical to use interpretations to show that an argument is valid in FOL, or that a sentence is a logical truth
- Instead, we need to use a different method: *formal proofs!*

Building on TFL Proofs

- This week and next, we will look at how to construct proofs in FOL
- When proving things in FOL, we will use **all** of the rules that we used in TFL
 - That includes basic **and derived** rules!
- All we need to do is add some **extra rules** to the system
 - This week we will add the basic rules for the quantifiers
 - Next week we will add some extra derived rules, plus the basic rules for identity

Introduction and Elimination

- Just like the connectives of TFL, each quantifier is governed by an **Introduction Rule** and an **Elimination Rule**
- Annoyingly, both quantifiers have an easy rule and a hard rule
 - The Introduction Rule for \exists is easy, but the Elimination Rule is hard
 - The Elimination Rule for \forall is easy, but the Introduction Rule is hard
- We will start with the easy rules, and then look at the harder ones later

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What can You Infer from a Universal Generalisation?

- Suppose you knew that the following universal generalisation is true:
 - Everyone loves *Intermediate Logic*
- You could then infer that this generalisation holds of each individual person
 - Tereza loves *Intermediate Logic*
 - Ben loves *Intermediate Logic*
 - Lucy loves *Intermediate Logic*
 - ...
- This leads us to our Universal Elimination Rule

Universal Elimination

$$\begin{array}{l|l}
 m & \forall x \mathcal{A}(\dots x \dots x \dots) \\
 & \mathcal{A}(\dots c \dots c \dots)
 \end{array}
 \quad \forall E, m$$

- $\mathcal{A}(\dots x \dots x \dots)$ is a formula containing **one or more** occurrences of some variable x
- c can be any name you like
- $\mathcal{A}(\dots c \dots c \dots)$ is the result of replacing **all** of the occurrences of x in $\mathcal{A}(\dots x \dots x \dots)$ with c

Some Examples

8		$\forall xRax$	
...		...	
15		Rab	$\forall E, 8$

Some Examples

8		$\forall yRay$	
...		...	
15		Rab	$\forall E, 8$

Some Examples

8		$\forall yRay$	
...		...	
15		Rac	$\forall E, 8$

Some Examples

8		$\forall yRay$	
...		...	
15		Raa	$\forall E, 8$

Some Examples

8		$\forall x(Fx \rightarrow (Rax \vee Gb))$	
...		...	
15		$Fa \rightarrow (Raa \vee Gb)$	$\forall E, 8$

Some Examples

8		$\forall x(Fx \rightarrow (Rax \vee Gb))$	
...		...	
15		$Fb \rightarrow (Rab \vee Gb)$	$\forall E, 8$

Some Examples

8		$\forall x(Fx \rightarrow (Rax \vee Gb))$	
...		...	
15		$Fc \rightarrow (Rac \vee Gb)$	$\forall E, 8$

A *Bad* Example

8	$\forall x(Fx \rightarrow (Rax \vee Gb))$	
...	...	
15	$Fc \rightarrow (Rax \vee Gb)$	$\forall E, 8$

- This is a *bad* example of Universal Elimination, because we replaced some but **not all** of the 'x's with 'c's

Two Universal Eliminations

1		$\forall x \forall y Rxy$	
		<hr/>	
2		$\forall y Ray$	$\forall E, 1$
3		Rab	$\forall E, 2$

Not One Double Elimination!

$$\begin{array}{l|l} 1 & \forall x \forall y Rxy \\ \hline 2 & Rab \end{array} \quad \forall E, 1$$

Two Universal Eliminations!!!

1		$\forall x \forall y Rxy$	
		<hr/>	
2		$\forall y Ray$	$\forall E, 1$
3		Rab	$\forall E, 2$

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How would You Prove an Existential Generalisation?

- Imagine you wanted to prove that an existential generalisation was true, for example:
 - Someone loves *Intermediate Logic*
- A really excellent way of doing this would be by first proving that this generalisation is true of some particular person
 - If you first proved that Nick loves *Intermediate Logic*, you could then infer that *someone* loves *Intermediate Logic*
- This motivates our Existential Introduction rule

Existential Introduction

$$\begin{array}{l|l}
 m & \mathcal{A}(\dots c \dots c \dots) \\
 & \exists \chi \mathcal{A}(\dots \chi \dots c \dots) \quad \exists I, m
 \end{array}$$

- $\mathcal{A}(\dots c \dots c \dots)$ is a sentence containing **one or more** occurrences of the name c
- χ can be any variable that does **not** occur in $\mathcal{A}(\dots c \dots c \dots)$
- $\mathcal{A}(\dots \chi \dots c \dots)$ is the result of replacing **one or more** of the occurrences of c in $\mathcal{A}(\dots c \dots c \dots)$ with the variable χ

Some Examples

8		$Raba$	
...		...	
15		$\exists x Rxba$	$\exists I, 8$

Some Examples

8		$Raba$	
...		...	
15		$\exists x Rabx$	$\exists I, 8$

Some Examples

8		$Raba$	
...		...	
15		$\exists x Rxbx$	$\exists I, 8$

Some Examples

8		<i>Raba</i>	
...		...	
15		$\exists y Ryby$	$\exists I, 8$

Some Examples

8		$Raba$	
...		...	
15		$\exists z Rzbz$	$\exists I, 8$

Some Examples

8		$Pa \rightarrow (Fb \vee \neg Sac)$	
...		...	
15		$\exists x(Px \rightarrow (Fb \vee \neg Sac))$	$\exists I, 8$

Some Examples

8		$Pa \rightarrow (Fb \vee \neg Sac)$	
...		...	
15		$\exists x(Pa \rightarrow (Fb \vee \neg Sxc))$	$\exists I, 8$

Some Examples

8		$Pa \rightarrow (Fb \vee \neg Sac)$	
...		...	
15		$\exists x(Px \rightarrow (Fb \vee \neg Sxc))$	$\exists I, 8$

Some Examples

8		$Pa \rightarrow (Fb \vee \neg Sac)$	
...		...	
15		$\exists y(Py \rightarrow (Fb \vee \neg Syc))$	$\exists I, 8$

Some Examples

8		$Pa \rightarrow (Fb \vee \neg Sac)$	
...		...	
15		$\exists z(Pz \rightarrow (Fb \vee \neg Szc))$	$\exists I, 8$

A *Bad* Example

8	$Pa \rightarrow (Fb \vee \neg Sac)$	
...	...	
15	$\exists z(Pz \rightarrow (Fz \vee \neg Szc))$	$\exists I, 8$

- This is a *bad* example of Existential Introduction, because we replaced **two different names** ('*a*' and '*b*') with the same variable

Two Existential Introductions

1		Rab	
		└──	
2		$\exists yRay$	$\exists I, 1$
3		$\exists x\exists yRxy$	$\exists I, 2$

Not One Double Introduction!

$$\begin{array}{l|l} 1 & Rab \\ \hline 2 & \exists x\exists yRxy \quad \exists I, 1 \end{array}$$

Two Existential Introductions!!!

1		Rab	
		<hr/>	
2		$\exists yRay$	$\exists I, 1$
3		$\exists x\exists yRxy$	$\exists I, 2$

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How would You Prove a Universal Generalisation?

- Imagine you wanted to prove that a universal generalisation was true, for example:
 - Everyone loves *Intermediate Logic*
- One idea would be to go through everyone in the domain, and prove that the generalisation is true of each of them:
 - James loves *Intermediate Logic*
 - Camilla loves *Intermediate Logic*
 - Luke loves *Intermediate Logic*
 - ...
- That is a bit longwinded and impractical, but it works well enough in this case, since we are working with a finite domain

How would You Prove a Universal Generalisation?

- But now suppose you wanted to prove a universal generalisations about **infinitely** many things
 - Every number is either odd or even
- You definitely couldn't do **that** by going through all of the numbers one by one!
- But there is another way:
 - Start by letting a be an arbitrary number
 - Then prove of a that it is either odd or even
 - Then conclude that since a was just an arbitrarily chosen number, **every** number must be odd or even
- This leads us to our Universal Introduction Rule

Universal Introduction

$$\begin{array}{l|l}
 m & \mathcal{A}(\dots c \dots c \dots) \\
 & \forall \chi \mathcal{A}(\dots \chi \dots \chi \dots) \quad \forall I, m
 \end{array}$$

- $\mathcal{A}(\dots c \dots c \dots)$ is a sentence containing one or more occurrences of the name c , and $\mathcal{A}(\dots \chi \dots \chi \dots)$ is the formula that you get when you replace **all** of those occurrences of c with the variable χ
- c **must not** occur in any undischarged assumptions above line m (including the premises of the argument)
- c **must not** occur in $\forall \chi \mathcal{A}(\dots \chi \dots \chi \dots)$

A Good Example

1			Pa	
			┌	
2			Pa	R, 1
			└	
3		$Pa \rightarrow Pa$		$\rightarrow I, 1-2$
4		$\forall x(Px \rightarrow Px)$		$\forall I, 3$

A Bad Example

1	$\forall x Rxa$	
2	Raa	$\forall E, 1$
3	$\forall x Rxx$	$\forall I, 2$

- This is a bad argument because 'a' appeared in an undischarged assumption (line 1)
- In this case, we made a background assumption about a , and so a isn't really an **arbitrary** object!

Another Bad Example

1		$\forall x Rxx$	
		<hr/>	
2		Raa	$\forall E, 1$
3		$\forall y Ray$	$\forall I, 2$

- This is a bad argument because we only replaced **some** occurrences of 'a' with 'y'

A Good Example (Again!)

1	$\forall x Rxx$	
2	Raa	$\forall E, 1$
3	$\forall y Ryy$	$\forall I, 2$

- This is a bad argument because we only replaced **some** occurrences of 'a' with 'y'
- If we replaced **all** of the occurrences of 'a' with 'y', the inference would've been trivial, but fine

Two Universal Introductions

1	Fa	
	───	
2	Fa	R, 1
3	$Fa \rightarrow Fa$	$\rightarrow I$, 1–2
4	Gb	
	───	
5	Gb	R, 4
6	$Gb \rightarrow Gb$	$\rightarrow I$, 4–5
7	$(Fa \rightarrow Fa) \wedge (Gb \rightarrow Gb)$	$\wedge I$, 3, 6
8	$\forall y((Fa \rightarrow Fa) \wedge (Gy \rightarrow Gy))$	$\forall I$, 7
9	$\forall x \forall y((Fx \rightarrow Fx) \wedge (Gy \rightarrow Gy))$	$\forall I$, 8

Not One Double Introduction!

1		Fa	
		—	
2		Fa	R, 1
3		$Fa \rightarrow Fa$	$\rightarrow I$, 1–2
4		Gb	
		—	
5		Gb	R, 4
6		$Gb \rightarrow Gb$	$\rightarrow I$, 4–5
7		$(Fa \rightarrow Fa) \wedge (Gb \rightarrow Gb)$	$\wedge I$, 3, 6
8		$\forall x \forall y ((Fx \rightarrow Fx) \wedge (Gy \rightarrow Gy))$	$\forall I$, 7

Two Universal Introductions!!!

1	Fa	
	───	
2	Fa	R, 1
3	$Fa \rightarrow Fa$	$\rightarrow I$, 1–2
4	Gb	
	───	
5	Gb	R, 4
6	$Gb \rightarrow Gb$	$\rightarrow I$, 4–5
7	$(Fa \rightarrow Fa) \wedge (Gb \rightarrow Gb)$	$\wedge I$, 3, 6
8	$\forall y((Fa \rightarrow Fa) \wedge (Gy \rightarrow Gy))$	$\forall I$, 7
9	$\forall x \forall y((Fx \rightarrow Fx) \wedge (Gy \rightarrow Gy))$	$\forall I$, 8

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What can You Infer from an Existential Generalisation?

- Suppose that you knew the following universal generalisation is true:
 - Someone loves *Intermediate Logic*
- You could not infer that that this generalisation holds of any particular person
 - It might be Becky who loves *Intermediate Logic*, or it might be David, or it might be Martha...
- So what could you infer?

What can You Infer from an Existential Generalisation?

- You could argue like this:
 - Suppose that Martha loves *Intermediate Logic*
 - Given that Martha loves *Intermediate Logic*, she must be attending all the lectures
 - The same would go for anyone else, if they loved *Intermediate Logic*
 - So even if I drop my supposition that Martha loves *Intermediate Logic*, since I do know that someone loves it, there must be someone who is attending all the lectures
- This motivates our Existential Elimination Rule

Existential Elimination

$$\begin{array}{l|l}
 m & \exists x \mathcal{A}(\dots x \dots x \dots) \\
 n & \left| \begin{array}{l} \mathcal{A}(\dots c \dots c \dots) \\ \hline \mathcal{B} \end{array} \right. \\
 o & \left| \mathcal{B} \right. \\
 & \mathcal{B} \qquad \qquad \qquad \exists E, m, n-o
 \end{array}$$

- c **must not** occur in any undischarged assumptions above line m (including the premises of the argument)
- c **must not** occur in $\exists x \mathcal{A}(\dots x \dots x \dots)$
- c **must not** appear in \mathcal{B}

Existential Elimination

$$\begin{array}{l|l}
 m & \exists x \mathcal{A}(\dots x \dots x \dots) \\
 n & \left| \begin{array}{l} \mathcal{A}(\dots c \dots c \dots) \\ \hline \mathcal{B} \end{array} \right. \\
 o & \mathcal{B}
 \end{array}
 \quad \exists E, m, n-o$$

- c **must not** appear in any line before m
- c **must not** appear in \mathcal{B}

An Example

- $\exists xRax, \forall y(Ray \rightarrow Fy) \therefore \exists zFz$

1	$\exists xRax$	
2	$\forall y(Ray \rightarrow Fy)$	
3	Rab	
4	$Rab \rightarrow Fb$	$\forall E, 2$
5	Fb	$\rightarrow E, 4, 3$
6	$\exists zFz$	$\exists I, 5$
7	$\exists zFz$	$\exists E, 1, 3-6$

A Bad Example!

1	$\exists x Rax$									
2	$\forall y (Ray \rightarrow Fy)$									
3	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">Rab</td> <td></td> </tr> <tr> <td colspan="2" style="border-top: 1px solid black; padding-top: 5px;"></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$Rab \rightarrow Fb$</td> <td style="padding-left: 10px;">$\forall E, 2$</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">Fb</td> <td style="padding-left: 10px;">$\rightarrow E, 4, 3$</td> </tr> </table>	Rab				$Rab \rightarrow Fb$	$\forall E, 2$	Fb	$\rightarrow E, 4, 3$	
Rab										
$Rab \rightarrow Fb$	$\forall E, 2$									
Fb	$\rightarrow E, 4, 3$									
6	Fb	$\exists E, 1, 3-5$								

- This is a bad argument because line 5 contains the name b , which is the name we introduced at line 3

Another Bad Example!

1	$\exists x Rax$	
2	$\forall y (Ray \rightarrow Fy)$	
3	Raa	
4	$Raa \rightarrow Fa$	$\forall E, 2$
5	Fa	$\rightarrow E, 4, 3$
6	$Fa \wedge Raa$	$\wedge I, 3, 5$
7	$\exists x (Fx \wedge Rxx)$	$\exists I, 6$
8	$\exists x (Fx \wedge Rxx)$	$\exists E, 1, 3-7$

- This is a bad proof, because the name we introduced at line 3 already appeared in lines 1 and 2

Two Existential Eliminations

1	$\exists x \exists y Rxy$	
2	$\forall x \forall y (Rxy \rightarrow Gy)$	
3	$\exists y Ray$	
4	Rab	
5	$\forall y (Ray \rightarrow Gy)$	$\forall E, 2$
6	$Rab \rightarrow Gb$	$\forall E, 5$
7	Gb	$\rightarrow E, 6, 4$
8	$\exists x Gx$	$\exists I, 7$
9	$\exists x Gx$	$\exists E, 3, 4-8$
10	$\exists x Gx$	$\exists E, 1, 3-9$

Not One Double Elimination!

1	$\exists x \exists y Rxy$	
2	$\forall x \forall y (Rxy \rightarrow Gy)$	
3	Rab	
4	$\forall y (Ray \rightarrow Gy)$	$\forall E, 2$
5	$Rab \rightarrow Gb$	$\forall E, 4$
6	Gb	$\rightarrow E, 5, 3$
7	$\exists x Gx$	$\exists I, 6$
8	$\exists x Gx$	$\exists E, 1, 3-7$

Two Existential Eliminations!!!

1	$\exists x \exists y Rxy$	
2	$\forall x \forall y (Rxy \rightarrow Gy)$	
3	$\exists y Ray$	
4	Rab	
5	$\forall y (Ray \rightarrow Gy)$	$\forall E, 2$
6	$Rab \rightarrow Gb$	$\forall E, 5$
7	Gb	$\rightarrow E, 6, 4$
8	$\exists x Gx$	$\exists I, 7$
9	$\exists x Gx$	$\exists E, 3, 4-8$
10	$\exists x Gx$	$\exists E, 1, 3-9$

Re-Introducing the Single Turnstile

- We will continue to use the single turnstile, ' \vdash ' to express provability
 - It is possible to construct a proof which starts with $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ as premises, and ends with \mathcal{C} as the conclusion
 - $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \vdash \mathcal{C}$
 - It is possible to construct a proof which doesn't have any premises, and ends with \mathcal{C} as the conclusion
 - $\vdash \mathcal{C}$
- But now we can use the rules for quantifiers as well as all the rules you learnt for TFL

Exercises

- Provide a proof for each of the following:
 1. $\vdash \forall z(Pz \vee \neg Pz)$
 2. $\forall x(Ax \rightarrow Bx), \exists xAx \vdash \exists xBx$
 3. $\forall x(Mx \leftrightarrow Nx), Ma \wedge \exists xRxa \vdash \exists xNx$
 4. $\forall x(\neg Mx \vee Ljx), \forall x(Bx \rightarrow Ljx), \forall x(Mx \vee Bx) \vdash \forall xLjx$
 5. $\forall x\forall yGxy \vdash \exists xGxx$