Intermediate Logic
Lecture Six

More on Interpretations

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More on Interpretations

Interpretations

- An **interpretation** is a specification of these three things:
  (1) The referent of each name we are dealing with
  (2) The extension of each predicate we are dealing with
  (3) The domain of quantification
- We can present an interpretation with a symbolisation key
- Or we can use the direct method, where we directly stipulate what the extension of each predicate will be, and what will be included in the domain

An Example of the Direct Method

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

How the Semantics Works

- A semantics for FOL is a machine for assigning truth-values to FOL sentences
  - We feed in an interpretation, and the semantics spits out truth-values
- There are three kinds of sentence to deal with:
  (i) Atomic sentences
  (ii) Sentences whose main logical operator is a sentential connective
  (iii) Sentences whose main logical operator is a quantifier

Atomic Sentences

- Let $\mathcal{R}^n$ be an n-place predicate, and a_1, a_2,..., a_n be names:
  - $\mathcal{R}^n a_1 a_2, \ldots, a_n$ is true in an interpretation iff $\mathcal{R}$ is true of the objects named by a_1, a_2,..., a_n in that interpretation (in that order)
- Let a and b be names:
  - $a = b$ is true in an interpretation iff a and b name the very same object in that interpretation

Sentential Connectives

- $\neg\mathcal{A}$ is true in an interpretation iff $\mathcal{A}$ is not true in that interpretation
- $\mathcal{A} \land \mathcal{B}$ is true in an interpretation iff $\mathcal{A}$ is true in that interpretation and $\mathcal{B}$ is true in that interpretation
- $\mathcal{A} \lor \mathcal{B}$ is true in an interpretation iff $\mathcal{A}$ is true in that interpretation or $\mathcal{B}$ is true in that interpretation (or both)
- $\mathcal{A} \to \mathcal{B}$ is true in an interpretation iff $\mathcal{A}$ is false in that interpretation or $\mathcal{B}$ is true in that interpretation (or both)
- $\mathcal{A} \leftrightarrow \mathcal{B}$ is true in an interpretation iff $\mathcal{A}$ and $\mathcal{B}$ have the same truth-value in that interpretation

Quantifiers

- Let c be a new name added to the language
- $\forall \chi \mathcal{A}(\ldots\chi\ldots\chi\ldots)$ is true in an interpretation iff $\mathcal{A}(\ldots c\ldots c\ldots)$ is true in *every* interpretation that extends the original interpretation by assigning an object to c (without changing the interpretation in any other way)
- $\exists \chi \mathcal{A}(\ldots\chi\ldots\chi\ldots)$ is true in an interpretation iff $\mathcal{A}(\ldots c\ldots c\ldots)$ is true in *some* interpretation that extends the original interpretation by assigning an object to c (without changing the interpretation in any other way)

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Fa

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Fa TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Fb

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Fb TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Ga

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Ga FALSE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$Fb \to Ga$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

$Fb \rightarrow Ga$ FALSE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

$Ha \leftrightarrow Ga$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$Ha \leftrightarrow Ga$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\forall xFx$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Fx

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Fc

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Fc	TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

Fc TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 2

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Fc TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\forall x Fx$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\forall x(Gx \lor Hx)$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

$$Gx \vee Hx$$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$$Gc \lor Hc$$

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$Gc \vee Hc$ FALSE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\forall x(Gx \vee Hx)$ FALSE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\forall x(Hx \rightarrow Gx)$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$$Hx \rightarrow Gx$$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$$Hc \rightarrow Gc$$

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

$Hc \to Gc$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0,1\rangle$, $\langle 1,0\rangle$, $\langle 2,1\rangle$

$Hc \rightarrow Gc$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 2

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

$Hc \rightarrow Gc$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\forall x(Hx \rightarrow Gx)$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\exists y(Fy \wedge Gy)$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$$Fy \land Gy$$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$$Fc \wedge Gc$$

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$Fc \wedge Gc$ FALSE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$Fc \wedge Gc$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\exists y(Fy \wedge Gy)$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

$\forall x \exists y Rxy$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\exists y Rxy$

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$$\exists y Rcy$$

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$$\exists y Rcy$$

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Rcy

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Rcd

Examples

domain: 0, 1, 2

a: 0 *d*: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Rcd FALSE

Examples

domain: 0, 1, 2

a: 0 d: 1

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

Rcd TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

$\exists y Rcy$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$$\exists y Rcy$$

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Rcd

Examples

domain: 0, 1, 2

a: 0 *d*: 0

b: 1

c: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Rcd TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

$\exists yRcy$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 2

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$$\exists y Rcy$$

Examples

domain: 0, 1, 2

a: 0 d: 1

b: 1

c: 2

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Rcd TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 2

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\exists yRcy$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\forall x \exists y Rxy$ TRUE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\exists y \forall x Rxy$

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\forall x Rxc$

Examples

domain: 0, 1, 2

a: 0 d: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Rdc FALSE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 0

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\forall x Rxc$ FALSE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

$$\forall x Rxc$$

Examples

domain: 0, 1, 2

a: 0 d: 1

b: 1

c: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Rdc FALSE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$\forall x Rxc$ FALSE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 2

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

$$\forall x Rxc$$

Examples

domain: 0, 1, 2

a: 0 *d*: 2

b: 1

c: 2

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1\rangle$, $\langle 1, 0\rangle$, $\langle 2, 1\rangle$

Rdc FALSE

Examples

domain: 0, 1, 2

a: 0

b: 1

c: 2

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

$\forall x Rxc$ FALSE

Examples

domain: 0, 1, 2

a: 0

b: 1

F: 0, 1, 2

G: 1, 2

H^1:

R: $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle 2, 1 \rangle$

$\exists y \forall x Rxy$ FALSE

Exercises

- Consider the following interpretation:

  domain: 0, 1
  h: 1
  A: 0, 1
  N: 0
  S:$\langle 1, 0\rangle$

- What is the truth-value of the following sentences on this interpretation?
  1. $Ah \wedge Nh$
  2. $\forall y Ay$
  3. $\exists x(Ax \wedge Nx)$
  4. $\forall x(Shx \rightarrow Nx)$
  5. $\exists x \forall y(Sxy \leftrightarrow Ny)$
  6. $\forall x \exists y(Ax \wedge Ny)$

Exercises

- Consider the following interpretation:

  domain: 0, 1
  h: 1
  A: 0, 1
  N: 0
  S:$\langle 1, 0 \rangle$

- What is the truth-value of the following sentences on this interpretation?
  1. $Ah \wedge Nh$
  2. $\forall y Ay$
  3. $\exists x(Ax \wedge Nx)$
  4. $\forall x(Shx \rightarrow Nx)$
  5. $\exists x \forall y(Sxy \leftrightarrow Ny)$
  6. $\forall x \exists y(Ax \wedge Ny)$

Exercises!!!

- For each list of sentences, provide one interpretation which makes them all true:

1. Fb, $\neg Gb$, $\exists x Gx$
2. Rab, $\exists x(Rax \wedge Gx)$
3. $\exists x \exists y(\neg x = y \wedge (Fx \wedge Gy))$, $\forall x(Fx \rightarrow Gx)$
4. $\neg \exists x(Fx \wedge Gx)$, Fa, Gb
5. Rab, $\forall x \forall y(Rxy \rightarrow Ryx)$
6. $\forall x \exists y Rxy$, $\neg \exists y \forall x Rxy$
7. Rab, $\forall x \forall y(Rxy \rightarrow Ryx)$, $\neg \exists x \exists y(\neg x = y \wedge (Rxy \wedge Ryx))$
8. Fb, $\forall y(Fy \rightarrow y = a)$
9. $\exists x(Fx \wedge \forall y(Fy \rightarrow y = x) \wedge Rxb)$

More on Interpretations

Logical Concepts

- Right at the beginning of this module, we defined a number of key logical ideas in terms of **possible worlds**
- A sentence is **necessarily true** iff it is true in every possible world
- A collection of sentences are **jointly consistent** iff they are all true together in some possible world
- An argument is **valid** iff there is no possible world in which all of its premises are true and its conclusion is false

From Possible Worlds to Valuations

- These definitions are intuitive, and are great for some informal purposes, but they are not much use for us
  - The whole idea of a possible world is a little bit wooly, and it would be better if we could replace it with something more precise
- Back in Lecture 1, we saw that when we are dealing with TFL, we can swap possible worlds for **valuations**
- A sentence is a **tautology** iff it is true on every valuation
- The sentences $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ **tautologically entail** the sentence $\mathcal{C}$ if there is no valuation on which all of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ true and $\mathcal{C}$ is false

From Valuations to Interpretations

- These definitions in terms of valuations are great for TFL, but they are no use when we are dealing with FOL
- But we can still offer similar definitions of the key logical ideas
- All we need to do is swap valuations for **interpretations**

The Key Logical Ideas

- $\mathcal{A}$ is a **logical truth** iff $\mathcal{A}$ is true in every interpretation
- $\mathcal{A}$ is a **contradiction** iff $\mathcal{A}$ is false in every interpretation
- $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \therefore \mathcal{C}$ is **valid in FOL** iff there is no interpretation in which $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are all true and $\mathcal{C}$ is false
- $\mathcal{A}$ and $\mathcal{B}$ are **logically equivalent** iff they are true in exactly the same interpretations
- $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are **jointly consistent** iff there is some interpretation in which all of the sentences are true

The Double-Turnstile, $\vDash$

- We will use '$\vDash$' for FOL much as we did for TFL:
  - There is no interpretation in which $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are all true and $\mathcal{C}$ is false
  - $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash \mathcal{C}$

  - $\mathcal{A}$ is true in every interpretation
  - $\vDash \mathcal{A}$

- I hope that by now, I do not need to bang on too much about how important it is not to confuse '$\vDash$' with '$\rightarrow$'!

More on Interpretations

Not a Logical Truth

- Suppose you wanted to show that '$\exists x Px \rightarrow Pa$' is **not** a logical truth
- This would require showing that this sentence is not true in **every** interpretation
- The best way of doing that is by cooking up an interpretation on which it is false:

  domain: people born before 2000CE
  a: Bertrand Russell
  P: ___ is German

- '$\exists x Px \rightarrow Pa$' is false in this interpretation
  - '$\exists x Px$' is true in this interpretation
  - 'Pa' is false in this interpretation

Not a Contradiction

- Now suppose you wanted to show that '$\exists x Px \rightarrow Pa$' is **not** a contradiction
- This would require showing that this sentence is not false in **every** interpretation
- The best way of doing that is by cooking up an interpretation on which it is true:

  domain: people born before 2000CE
  a: Gottlob Frege
  P: ___ is German

- '$\exists x Px \rightarrow Pa$' is true in this interpretation
  - '$\exists x Px$' is true in this interpretation
  - 'Pa' is true in this interpretation

Jointly Consistent

- Now imagine that you wanted to show that the following sentences are *jointly* consistent:
  - $\forall x(Fx \to Gx), \neg\forall x Gx$
- You would need to cook up an interpretation in which both of these sentences are true

  domain: people born before 2000CE
  F: __ is less than 10 years old
  G: __ is German

- '$\forall x(Fx \to Gx)$' is true in this interpretation
  - 'F' is not true of anything in the domain, and so '$Fa \to Ga$' is true no matter who in the domain we use 'a' to name
- '$\neg\forall x Gx$' is also true in this interpretation
  - 'G' is not true of everything in the domain

Not Logically Equivalent

- Now imagine that you wanted to show that '$\exists x(Fx \wedge Gx)$' and '$\exists x(Fx \rightarrow Gx)$' are **not** logically equivalent
- This would require showing that there is some interpretation which makes one of them true and the other false

  domain: people born before 2000CE
  F: ___ is younger than 10 years old
  G: ___ is German

- '$\exists x(Fx \wedge Gx)$' is false in this interpretation
  - 'F' is not true of anything in the domain, so '$Fa \wedge Ga$' would be false, no matter who in the domain we use 'a' to name
- But '$\exists x(Fx \rightarrow Gx)$' is true in this interpretation
  - 'F' is not true of anything, so '$Fa \rightarrow Ga$' is guaranteed to be true, no matter who in the domain we use 'a' to name

Not Valid in FOL

- Lastly, imagine that you wanted to show that the following argument is **not** valid in FOL:
  - $\forall x \exists y Rxy \therefore \exists y \forall x Rxy$
- You would need to come up with an interpretation which makes the premise true and the conclusion false

  domain: people born before 2000CE
  R: $\_\__1$ is a child of $\_\__2$

- '$\forall x \exists y Rxy$' is true in this interpretation
  - Everyone born before 2000CE is a child of someone born before 2000CE
- '$\exists y \forall x Rxy$' is false in this interpretation
  - It is not the case that there is someone born before 2000CE who is a parent of everyone born before 2000CE

Summing Up

- To show that $\mathcal{A}$ is **not** a logical truth, construct an interpretation in which $\mathcal{A}$ is false
- To show that $\mathcal{A}$ is **not** a contradiction, construct an interpretation in which $\mathcal{A}$ is true
- To show that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ **are** jointly consistent, construct an interpretation in which $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are all true
- To show that $\mathcal{A}$ is **not** logically equivalent to $\mathcal{B}$, construct an interpretation in which $\mathcal{A}$ and $\mathcal{B}$ have different truth-values
- To show that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \therefore \mathcal{C}$ is **not** valid in FOL, construct an interpretation in which $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are all true but $\mathcal{C}$ is false

Counter-Interpretations

- Suppose you constructed an interpretation in which $\mathcal{A}$ is false
- We would call that a **counter-interpretation** to the claim that $\mathcal{A}$ is a logical truth

- Suppose that you constructed an interpretation in which $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are all true but $\mathcal{C}$ is false
- We would call that a **counter-interpretation** to the claim that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \therefore \mathcal{C}$ is valid in FOL

- Suppose that you constructed an interpretation in which $\mathcal{A}$ and $\mathcal{B}$ have different truth-values
- We would call that a **counter-interpretation** to the claim that $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent

Counter-Interpretations

- Suppose you constructed an interpretation in which $\mathcal{A}$ is false
- We would call that a **counter-interpretation** to the claim that $\vDash \mathcal{A}$

- Suppose that you constructed an interpretation in which $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are all true but $\mathcal{C}$ is false
- We would call that a **counter-interpretation** to the claim that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \therefore \mathcal{C}$ is valid in FOL

- Suppose that you constructed an interpretation in which $\mathcal{A}$ and $\mathcal{B}$ have different truth-values
- We would call that a **counter-interpretation** to the claim that $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent

Counter-Interpretations

- Suppose you constructed an interpretation in which $\mathcal{A}$ is false
- We would call that a **counter-interpretation** to the claim that $\vDash \mathcal{A}$

- Suppose that you constructed an interpretation in which $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are all true but $\mathcal{C}$ is false
- We would call that a **counter-interpretation** to the claim that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash \mathcal{C}$

- Suppose that you constructed an interpretation in which $\mathcal{A}$ and $\mathcal{B}$ have different truth-values
- We would call that a **counter-interpretation** to the claim that $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent

Counter-Interpretations

- Suppose you constructed an interpretation in which $\mathcal{A}$ is false
- We would call that a **counter-interpretation** to the claim that $\vDash \mathcal{A}$

- Suppose that you constructed an interpretation in which $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are all true but $\mathcal{C}$ is false
- We would call that a **counter-interpretation** to the claim that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash \mathcal{C}$

- Suppose that you constructed an interpretation in which $\mathcal{A}$ and $\mathcal{B}$ have different truth-values
- We would call that a **counter-interpretation** to the claim that both $\mathcal{A} \vDash \mathcal{B}$ and $\mathcal{B} \vDash \mathcal{A}$

Exercises

- Present interpretations which show that the following are false:
    1. $\exists x(Px \rightarrow Qx) \vDash \exists xPx$
    2. $Na \wedge Nb \wedge Nc \vDash \forall xNx$
    3. $\exists x(Jx \wedge Kx), \exists x \neg Kx, \exists x \neg Jx \vDash \exists x(\neg Jx \wedge \neg Kx)$
    4. $Lab \rightarrow \forall xLxb, \exists xLxb \vDash Lbb$
    5. $\forall x(Dx \rightarrow \exists yTyx) \vDash \exists y \exists z \neg y = z$

Exercises!!!!

- Present interpretations which show that the following are false:
    1. $\vDash \forall x Px \lor \forall x \neg Px$
    2. $\vDash (\exists x Hx \land \exists x Jx) \to \exists x(Hx \land Jx)$
    3. $\vDash \forall x Fx \to \exists x Fx$
    4. $\vDash \forall x(Fx \to Gx) \to \exists x Gx$