Intermediate Logic
Lecture Four

First-Order Logic

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First-Order Logic

Introducing First-Order Logic

Names

Predicates

Quantifiers

Putting It All Together

Identity

The Limits of TFL

- Consider the following obviously valid argument:
  - Sharon studies archaeology
  - Everyone who studies archaeology wishes that they were Indiana Jones
  - So Sharon wishes that she were Indiana Jones
- We cannot use TFL to show that this argument is valid
- The trouble is that, as far as TFL is concerned, the three sentences are all just atoms

  A: Sharon studies archaeology

  B: Everyone who studies archaeology wishes that they were Indiana Jones

  C: Sharon wishes that she were Indiana Jones

Splitting the (Logical) Atom

- To improve on TFL, we need to find a way of breaking atomic sentences down into subatomic units
  - An **atom** is a sentence which is not built out of any smaller *sentences*
  - In TFL, atoms have absolutely no internal structure
  - What we need is a logical system in which atomic sentences are built out of smaller *sub-sentential* expressions
- The system which does this is known as **First-Order Logic** (FOL)
  - This is the system you called Predicate Logic
  - We are calling it 'First-Order Logic' because there is another kind of predicate logic out there, called 'Second-Order Logic'

The Three Basic Building Blocks

- **Names**
  - Names in English: 'Gottlob Frege', 'Ludwig Wittgenstein', 'Rob Trueman'
  - Names in FOL: 'a', 'b', 'c', ... 'r'

- **Predicates**
  - Predicates in English: '__ is wise', '__ is human', '__ is a dog'
  - Predicates in FOL: 'A', 'B', 'C'...

- **Quantifiers**
  - Quantifiers in English: 'Everything', 'Something'
  - Quantifiers in FOL: '$\forall$', '$\exists$'

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Predicates

Quantifiers

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Names versus Singular Terms

- In general, a **singular term** is an expression which stands for a specific person, place or thing
  - 'Bertrand Russell' stands for a specific person, Bertrand Russell
  - 'The inventor of quantified logic' stands for a specific person, Gottlob Frege
- These two expressions are quite different:
  - 'Bertrand Russell' is a *proper name*; it's job is just to stand for Bertrand Russell
  - 'The inventor of quantified logic' is a *definite description*; it's job is to pick out whoever satisfies that description
- The **names** in FOL are meant to be symbolisations of proper names, not definite descriptions
  - We'll come back to definite descriptions later!

Names in FOL

- Names in FOL are lower case letters between 'a' and 'r', and if we want even more names, then we can add numerical subscritps (e.g. 'q_{27}')
- Each name stands for **exactly one** thing
  - There are no *ambiguous* names which sometimes refer to one thing, sometimes to another
  - However, there is nothing wrong with **one** object being referred to by **two** (or more!) names
- When we provide a symbolisation key for FOL, here is how we specify what each name refers to:

  b: Bertrand Russell
  f: Gottlob Frege

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Introducing First-Order Logic

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Quantifiers

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English Predicates

- The simplest predicates in English are expression which attribute properties to individuals; they let us say things about objects

- Here's an example of an English predicate: '___ is wise'
  - '___ is wise' attributes the property of wisdom
  - '___ is wise' says of an individual that they are wise

- Here's another example: '___ loves *Intermediate Logic*'
  - '___ loves *Intermediate Logic*' attributes the property of loving *Intermediate Logic*
  - '___ loves *Intermediate Logic*' says of an individual that they love *Intermediate Logic*

How to Make Predicates

- In general, you can think of predicates as things which combine with singular terms to make sentences
  - When you combine the predicate '__ is wise' with the name 'Socrates', you get the sentence 'Socrates is wise'

- Alternatively, you can think of a predicate as what you get when you remove a singular term from a sentence
  - Start with the sentence 'Daniel stole the ball from Simon'
  - If you remove 'Daniel', then you get: '__ stole the ball from Simon'
  - If you remove 'the ball', then you get: 'Daniel stole __ from Simon'
  - If you remove 'Simon', then you get: 'Daniel stole the ball from __'

Predicates of Higher Adicities

- The predicates that we have been looking at so far are all **monadic** predicate, meaning that they combine with just one name at a time
  - '___ is wise' has one gap for a name to be plugged into
- But other predicates combine with *more than one* name at a time
  - **Dyadic predicates** combine with *two* names at a time, e.g. '___ loves ___'
  - **Triadic predicates** combine with *three* names at a time, e.g. '___ is between ___ and ___'
- We call the number of names that a predicate can combine with its **adicity**, and you can have predicates of any adicity whatsoever

Predicates in FOL

- Predicates in FOL are capital letters, and we can add numerical subscripts if we ever need more than 26 predicates (e.g. 'V_{342}')

- We also really need some way of indicating the adicity of each predicate; we will do that with numerical superscripts:

  - **Monadic predicates:**
    $A^1, B^1, \ldots, Z^1,\ A^1_1, B^1_1, \ldots, Z^1_1,\ A^1_2, B^1_2, \ldots, Z^1_2,\ \ldots$

  - **Dyadic predicates:**
    $A^2, B^2, \ldots, Z^2,\ A^2_1, B^2_1, \ldots, Z^2_1,\ A^2_2, B^2_2, \ldots, Z^2_2,\ \ldots$

  - ***n*-adic predicates:**
    $A^n, B^n, \ldots, Z^n,\ A^n_1, B^n_1, \ldots, Z^n_1,\ A^n_2, B^n_2, \ldots, Z^n_2,\ \ldots$

Symbolisation Keys for Predicates

- When we provide a symbolisation key for FOL, here is how we specify what each monadic predicate symbolises:

  A^1: ___ is angry
  H^1: ___ is happy

- So if 'g' symbolises 'Gottlob Frege', then'A^1g' symbolises 'Gottlob Frege is angry', and 'H^1g' symbolises 'Gottlob Frege is happy'

- And if 'b' symbolises 'Bertrand Russell', then 'A^1b' symbolises 'Bertrand Russell is angry', and 'H^1b' symbolises 'Bertrand Russell is happy'

Symbolisation Keys for Predicates

- Here is how to provide a symbolisation key for a dyadic predicate:

  L^2: $\_\__1$ loves $\_\__2$

- The little subscript numerals attached to the blanks are there to tell us the *order* in which 'L^2' applies to individuals
  - On this key, 'L^2' applies to the lover first, and to the beloved second
  - So 'L^2bg' symbolises 'Bertrand Russell loves Gottlob Frege'

- Contrast 'L^2' with 'M^2' on the following key:

  M^2: $\_\__2$ loves $\_\__1$

  - On this key, 'M^2' applies to the *beloved* first and the *lover* second
  - So 'M^2bg' symbolises 'Gottlob Frege loves Bertrand Russell'

Symbolisation Keys for Predicates

- Here is how to provide a symbolisation key for a dyadic predicate:

  K^2: $\_\__1$ kicks $\_\__2$

- The little subscript numerals attached to the blanks are there to tell us the *order* in which 'K^2' applies to individuals
  - On this key, 'K^2' applies to the kicker first, and to the kicked second
  - So 'K^2bg' symbolises 'Bertrand Russell kicks Gottlob Frege'
- Contrast 'K^2' with 'N^2' on the following key:

  N^2: $\_\__2$ kicks $\_\__1$

  - On this key, 'N^2' applies to the *kicked* first and the *kicker* second
  - So 'N^2bg' symbolises 'Gottlob Frege kicks Bertrand Russell'

Symbolisation Keys for Predicates

- Here is how to provide a symbolisation key for a dyadic predicate:

  Q^2: $\_\__1$ quizzes $\_\__2$

- The little subscript numerals attached to the blanks are there to tell us the *order* in which 'Q^2' applies to individuals
  - On this key, 'Q^2' applies to the quizzer first, and to the quizzed second
  - So 'Q^2bg' symbolises 'Bertrand Russell quizzes Gottlob Frege'

- Contrast 'Q^2' with 'R^2' on the following key:

  R^2: $\_\__2$ quizzes $\_\__1$

  - On this key, 'R^2' applies to the *quizzed* first and the *quizzer* second
  - So 'R^2bg' symbolises 'Gottlob Frege quizzes Bertrand Russell'

Let's Get Rid of those Superscripts!

- Strictly speaking, we need the superscript on an FOL predicate to tell us what its adicity is
- But in practice, we can usually tell what the adicity of a predicate is just by looking at how we actually use it
  - If I write '*Rab*', then unless I've messed up, '*R*' must be a dyadic predicate
  - Equally, '*S*' must be a triadic predicate if you give it the following entry in a symbolisation key:
    S: $\_\__1$ sold $\_\__2$ to $\_\__3$
- So from now on, we won't bother with those ugly superscripts unless we *really* have to

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Introducing First-Order Logic

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Putting It All Together

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Quantifiers

- FOL has two basic **quantifiers**
  - The *existential* quantifier, '$\exists$', is the FOL for 'Something'
  - The *universal* quantifier, '$\forall$', is the FOL for 'Everything'
- A quantifier must always be followed by a **variable**
  - A variable is a lowercase letter from 's' to 'z', with subscripts if we need them (e.g. 'x_{3000}')
- Here is an example: '$\forall x Hx$'
  - If 'H' is our sybmolisation for '___ is happy', then '$\forall x Hx$' says that everyone is happy
  - You should think of the 'x' as a kind of placeholder: whoever we pick as x, x is happy
- If we wanted to say that *someone* was happy, we would write: '$\exists x Hx$'

Domains of Quantification

- Very often, when we use the quantifier ‘everyone’ in English, we do not literally mean **everyone** in the *whole world*
- Normally, we are quantifying over a particular, limited **domain** of quantification
- Roughly, the domain of quantification is the collection of things we are talking about
- If we wanted to talk about the people in York, then we would pick the people in York to be our domain

  domain: people in York

- The quantifiers only quantify over things in the domain, and all our names need to pick out things in the domain

Scope

- Like other logical expressions, quantifiers come with a **scope**
  (1) If everyone is a singer, then Rob is a singer
  (2) Everyone is such that, if they are a singer, then Rob is a singer

- (1) is true: *everyone* includes me, so if everyone is a singer then I am a singer

- (2) is false: I am not a singer but Susanne Sundfør is; so it is not true of Susanne Sundfør that if she is a singer, then I am a singer!

- We can capture the difference between these two sentences in FOL by giving the universal quantifier different scope
  (1′) $\forall x Sx \rightarrow Sr$
  (2′) $\forall x(Sx \rightarrow Sr)$

Multiple-Generality

- Questions of scope become even more important when we are dealing with sentences which contain more than one quantifier:
  (1) Everyone loves someone
  (2) Someone is loved by everyone
- (1) means that each person loves someone, but leaves it open that different people may love different people
- (2) means that there is a single person who everyone loves
- We can capture the difference between these two sentences in FOL by giving the quantifiers different scope
  (1′) $\forall x \exists y Lxy$
  (2′) $\exists y \forall x Lxy$

First-Order Logic

Introducing First-Order Logic

Names

Predicates

Quantifiers

Putting It All Together

Identity

An Example Symbolisation

domain: Everyone born after 1900

b: Bertrand Russell

g: Gottlob Frege

A: $\_\_$ is angry

R: $\_\__1$ respects $\_\__2$

M: $\_\__2$ loves $\_\__1$

An Example Symbolisation

domain: Everyone born after 1900

b: Bertrand Russell

g: Gottlob Frege

L: $\_\_$ is a logician

R: $\_\__1$ respects $\_\__2$

M: $\_\__2$ loves $\_\__1$

- Frege is angry, unless Russell respects him $\Rightarrow Ag \vee Rbg$
- Someone angry is loved by Frege $\Rightarrow \exists x(Ax \wedge Mxg)$
- Everyone is loved by someone $\Rightarrow \forall x \exists y(Myx)$

Reading FOL

- **YOU WILL NOT BE ASSESSED ON YOUR ABILITY TO SYMBOLISE ENGLISH SENTENCES INTO FOL!**
- However, it *can* be helpful to know how to translate a sentence of FOL into English
- In preparation for this module, you should do all of the formalisation exercises in *forall* χ, but for now we will do some translations from FOL to English

Top Translation Tips

- $\forall\chi(\mathcal{A}\chi \to \mathcal{B}\chi)$ symbolises ‘All $\mathcal{A}$s are $\mathcal{B}$’ (or ‘Everything that is $\mathcal{A}$ is $\mathcal{B}$’)
- $\forall\chi(\mathcal{A}\chi \leftrightarrow \mathcal{B}\chi)$ symbolises ‘All $\mathcal{A}$s are $\mathcal{B}$, and all $\mathcal{B}$s are $\mathcal{A}$’
- $\exists\chi(\mathcal{A}\chi \wedge \mathcal{B}\chi)$ symbolises ‘Some $\mathcal{A}$ is $\mathcal{B}$’ (or ‘Something is $\mathcal{A}$ and $\mathcal{B}$’)

Top Translation Tips

- Keep an eye on the scope of the quantifiers
  - '$\forall x(Fx \rightarrow Ga)$' means something very different from '$\forall x Fx \rightarrow Ga$'!
- Keep an eye on the order of the quantifiers
  - '$\forall x \exists y Rxy$' means something very different from '$\exists y \forall x Rxy$'

Top Translation Tips

- $\neg\exists\chi\mathcal{A}$ and $\forall\chi\neg\mathcal{A}$ can *both* be translated as ‘Nothing is $\mathcal{A}$’
- $\neg\forall\chi\mathcal{A}$ and $\exists\chi\neg\mathcal{A}$ can *both* be translated as ‘Something is not $\mathcal{A}$’

Exercises

- Translate the following into English, using this key:

domain: Everyone born after 1900
d: David Attenborough
r: Richard Attenborough
A: ___ is an actor
Z: ___ is a zoologist
L: $\_\__1$ loves $\_\__2$

(i) $Ldr \wedge Lrd$
(ii) $\neg\exists x(Ax \wedge Zx)$
(ii) $\forall x(Zx \rightarrow Lxd)$
(iv) $\forall z\forall y(Az \rightarrow Lyz)$
(v) $\forall u\exists v(Av \wedge Luv)$
(vi) $\exists v\forall u(Zv \wedge Luv)$

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Introducing First-Order Logic

Names

Predicates

Quantifiers

Putting It All Together

Identity

A Limit of FOL So Far...

- Consider this English sentence:
  (1) Simon is mean to everyone
- On the face of it, it seems that we can easily symbolise this sentence:
  (2) $\forall x Msx$
- But (2) implies that Simon is mean to everyone, *including* Simon!
- That is not how we ordinarily hear (1): we normally take this to say that Simon is mean to everyone, *except* Simon
- But as it stands, FOL is unable to express this simple thought

Introducing Identity

- To deal with cases like this, we add an identity symbol to FOL

  $=$: $\_\__1$ is identical to $\_\__2$

- '$=$' is a dyadic predicate symbol, but unlike the other predicates it **has** to be used to express identity; we cannot change its meaning at any time

  (As a result, we don't need to bother including an entry for it in our symbolisation keys)

- Now return to this sentence:

  (1) Simon is mean to everyone

- We can symbolise it as:

  (2) $\forall x(\neg x = s \rightarrow Msx)$

There are at least...

- Consider these sentences:
  (1) There is at least one apple
  (2) There are at least two apples
  (3) There are at least three apples

- Now that we have '$=$', we can symbolise these sentences in FOL:
  (1′) $\exists x Ax$
  (2′) $\exists x \exists y(Ax \land Ay \land \neg x = y)$
  (3′) $\exists x \exists y \exists z(Ax \land Ay \land Az \land \neg x = y \land \neg y = z \land \neg z = x)$

There are at most...

- Consider these sentences:
  (1) There is at most one apple
  (2) There are at most two apples
- Now that we have '$=$', we can symbolise these sentences in FOL:
  (1′) $\neg\exists x \exists y(Ax \wedge Ay \wedge \neg x = y)$
  (2′) $\neg\exists x \exists y \exists z(Ax \wedge Ay \wedge Az \wedge \neg x = y \wedge \neg y = z \wedge \neg z = x)$

There are at most...

- Consider these sentences:
  (1) There is at most one apple
  (2) There are at most two apples
- Now that we have '$=$', we can symbolise these sentences in FOL:
  (1′) $\forall x \forall y((Ax \wedge Ay) \rightarrow x = y)$
  (2′) $\neg \exists x \exists y \exists z(Ax \wedge Ay \wedge Az \wedge \neg x = y \wedge \neg y = z \wedge \neg z = x)$

There are at most...

- Consider these sentences:
  (1) There is at most one apple
  (2) There are at most two apples
- Now that we have '$=$', we can symbolise these sentences in FOL:
  (1′) $\forall x \forall y((Ax \wedge Ay) \rightarrow x = y)$
  (2′) $\forall x \forall y \forall z((Ax \wedge Ay \wedge Az) \rightarrow (x = y \vee y = z \vee z = x))$

There are exactly...

- Consider this sentence:

  (1) There is exactly one apple

- (1) is the conjunction of these two sentences:

  (2) There is at least one apple

  (3) There is at most one apple

- So we can symbolise (1) in FOL as:

  (1′) $\exists x Ax \land \forall x \forall y((Ax \land Ay) \to x = y)$

There are exactly...

- Consider this sentence:

  (1) There is exactly one apple

- (1) is the conjunction of these two sentences:

  (2) There is at least one apple

  (3) There is at most one apple

- So we can symbolise (1) in FOL as:

  (1′) $\exists x(Ax \wedge \forall y(Ay \rightarrow x = y))$

There are exactly...

- Consider this sentence:
  (1) There is exactly one apple
- (1) is the conjunction of these two sentences:
  (2) There is at least one apple
  (3) There is at most one apple
- So we can symbolise (1) in FOL as:
  (1′) $\exists x \forall y (Ay \leftrightarrow x = y)$

Definite Descriptions

- Definite descriptions are expressions like ‘the *F*’
  - The inventor of quantified logic
  - The present Queen of England
  - The present King of France
- On the face of it, they look like singular terms, i.e. expressions which stand for objects
- But Russell famously insisted that they were not
- We will not now look at Russell’s reasons for this, but will just show how we can neatly formulate Russell’s approach to definite descriptions in FOL

Russell's Theory of Definite Descriptions

- The Queen of England is having lunch
  - (a) There is at least one queen of England; and
  - (b) There is at most one queen of England; and
  - (c) Every queen of England is having lunch
- The author of *Harry Potter* is very rich
  - (a) There is at least one author of *Harry Potter*; and
  - (b) There is at most one author of *Harry Potter*; and
  - (c) Anyone who authored *Harry Potter* is very rich

Russell's Theory of Definite Descriptions

- The F is G
  (a) There is at least one F; and
  (b) There is at most one F; and
  (c) All Fs are Gs
- In a short sentence:
  - There is exactly one F, and it is G
- In formal symbols:
  - $\exists x(Fx \wedge \forall y(Fy \rightarrow y = x) \wedge Gx)$
  - $\exists x(\forall y(Fy \leftrightarrow y = x) \wedge Gx)$

Practise!

- Translate the following into English, using this key:

domain: Everyone born after 1900
d: David Attenborough
r: Richard Attenborough
A: ___ is an actor
Z: ___ is a zoologist
L: $\_\__1$ loves $\_\__2$

(i) $\forall x(\neg x = r \rightarrow Lxd)$
(ii) $\exists x \exists y((Ax \wedge Ay) \wedge \neg x = y)$
(iii) $\exists x((Zx \wedge \forall y(Zy \rightarrow y = x)) \wedge Lxr)$
(iv) $\exists x \forall y((Zx \leftrightarrow y = x)) \wedge Lxr)$
(v) $\exists x \forall y(((Zx \wedge Lxr) \leftrightarrow y = x)$