Intermediate Logic
Lecture Two

Natural Deduction for TFL

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Natural Deduction for TFL

The Truth Table Test for Tautological Entailment

- Last week we looked at truth tables, which were largely familiar to you from *Reason and Argument*
- Truth tables give us a way of **testing** whether $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n \vDash C$
  - Draw up a truth table for $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n$ and C
  - Check whether there is any line of the truth table which makes $\mathcal{C}$ false and $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n$ all true
  - $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n \vDash C$ just in case there is no such line

The Truth Table Test can be Impractical

- This truth table test is great when you are dealing with just a few atoms, but things quickly get out of hand
- If you are dealing with n atoms, then you need to draw up a truth table with 2^n lines
  - 1 atom, 2 lines; 2 atoms, 4 lines; 3 atoms, 8 lines...
- This argument is *obviously* tautologically valid:
  - $A_1 \rightarrow C_1 \therefore (A_1 \wedge A_2 \wedge A_3 \wedge A_4 \wedge A_5 \wedge A_6 \wedge A_7 \wedge A_8 \wedge A_9 \wedge A_{10}) \rightarrow (C_1 \vee C_2 \vee C_3 \vee C_4 \vee C_5 \vee C_6 \vee C_7 \vee C_8 \vee C_9 \vee C_{10})$
- But to show that it is tautologically valid with a truth table, we would need to write out $2^{20} = 1,048,576$ lines!

Truth Tables are Static

- There is a sense in which the truth table test is **static**
  - You draw up the truth table for the premises and the conclusion, and you just search line by line for a valuation which makes all of the premises true and the conclusion false
- That is not how we really reason; real reasoning is **dynamic**
  - You make inferences from your premises, and move one step at a time towards your conclusion

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A Better Way

- In this lecture, we will move away from truth tables, and start looking at **formal proofs**
- In a formal proof, we show that an argument is valid by showing how we can move from the premises to the conclusion, by following clearly specified rules
- **This is a very different way of thinking about arguments!!!**
  - We no longer care about truth-values or valuations or anything like that
  - Constructing a proof is like playing a game, according to some carefully stated rules

Start with the Premises

- The starting point of any proof is the collection of premises
- Suppose we wanted to prove that the following argument is valid:
  - $A, A \rightarrow B \therefore A \wedge B$
- We start by writing out the premises, like this:

1	A
2	$A \rightarrow B$

Play by the Rules

- In a moment, we will begin introducing a bunch of rules
- These rules tell us how we are allowed to move from one sentence to another
- Once we have written out our premises, we are only allowed to write a new sentence if it is permitted by the rules
- Our aim is to get from the premises, via the rules, to the conclusion

End with the Conclusion

$A, A \to B \therefore A \land B$

1	A
2	$A \to B$
3	???
4	???
...	...
n	$A \land B$

What are the Rules?

- Now all you need to know to get proving things is what the rules for moving from one sentence to another actually are
- These rules break down into rules governing each sentential connective
- These rules come in two kinds
  - **Introduction Rules** allow you to introduce a connective into a sentence
  - **Elimination Rules** allow you to eliminate a connective from a sentence
- There is a complete list of these rules in *forallx*, and they are repeated in a handy appendix at the end of the book

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How would You Prove a Conjunction?

- Imagine you wanted to prove that a conjunction was true, for example:
  - Logic is interesting and logic is useful
- A really excellent way of doing this would be by first proving that each conjunct was true
  - If you proved that logic is interesting and also proved that logic is useful, you would be well within your rights to conclude that logic is interesting *and* logic is useful

Putting that in a Formal System

8	I	
...	...	
15	U	
16	$I \wedge U$	$\wedge$I, 8, 15

The Other Way Around

8	I	
...	...	
15	U	
16	$U \wedge I$	$\wedge$I, 15, 8

Conjunction Introduction

m	$\mathcal{A}$	
n	$\mathcal{B}$	
	$\mathcal{A} \wedge \mathcal{B}$	$\wedge$I, m, n

- This rule is called **Conjunction Introduction** because it introduces the conjunction symbol into our proof
- This rule is fully general and schematic: it doesn't matter what sentences $\mathcal{A}$ and $\mathcal{B}$ are, and it doesn't matter what numbers m and n are

What can You Infer from a Conjunction?

- Suppose you had proved a conjunction:
  - Logic is interesting and logic is useful
- What are you allowed to infer from this conjunction?
  - You can infer that logic is interesting
  - You can infer that logic is useful
- This motivates our **Conjunction Elimination** rules, which let us eliminate the conjunction symbol from our proof

Conjunction Elimination

m	$\mathcal{A} \wedge \mathcal{B}$	
	$\mathcal{A}$	$\wedge$E, m

m	$\mathcal{A} \wedge \mathcal{B}$	
	$\mathcal{B}$	$\wedge$E, m

A Simple Proof

$$(A \wedge B) \wedge C \therefore A \wedge (B \wedge C)$$

1	$(A \wedge B) \wedge C$	
2	$A \wedge B$	$\wedge$E, 1
3	C	$\wedge$E, 1
4	A	$\wedge$E, 2
5	B	$\wedge$E, 2
6	$B \wedge C$	$\wedge$I, 5, 3
7	$A \wedge (B \wedge C)$	$\wedge$I, 4, 6

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Modus Ponens

- You are all familiar with **modus ponens**: $\mathcal{A} \to \mathcal{B}, \mathcal{A} \therefore \mathcal{B}$
  - If Sharon studies archaeology, then Sharon tells Rob a lot about old pots; Sharon studies archaeology; therefore, Sharon tells Rob a lot about old pots
- In TFL, we just steal modus ponens, and make it our **elimination rule** for the conditional, $\to$

Conditional Elimination

m	$\mathcal{A} \to \mathcal{B}$	
n	$\mathcal{A}$	
	$\mathcal{B}$	$\to$E, m, n

How would You Prove a Conditional?

- How would you show that this argument is valid?
  - Sharon studies archaeology. Therefore, if Sharon understands Latin, then Sharon studies archaeology *and* understands Latin
- Here is a really good way of doing it:
  - Start by assuming the premise that Sharon studies archaeology
  - Now **additionally** assume that Sharon understands Latin
  - Then by conjunction introduction, Sharon studies archaeology and understands Latin
  - Of course, that's conditional on the extra assumption that Sharon understands Latin
  - But we can still infer that *if* Sharon understands Latin, then she both studies archaeology and understands Latin

Formalising this Argument

$A \therefore L \to (A \wedge L)$

1	A	
2	$\quad L$	
3	$\quad A \wedge L$	$\wedge$I, 1, 2
4	$L \to (A \wedge L)$	$\to$I, 2–3

Discharing Assumptions

1	A	
2	$\quad L$	
3	$\quad A \wedge L$	$\wedge$I, 1, 2
4	$L \rightarrow (A \wedge L)$	$\rightarrow$I, 2–3

- At line 4, we dropped back to using one vertical line
- This is to indicate that we have **discharged** the additional assumption 'L' (line 2)
- When we discharge an assumption, that assumption does not hang around as a premise of our argument
- It was just a temporary assumption made during the course of the argument, but the argument as a whole does not rely on it

Conditional Introduction

m	$\quad\mid \mathcal{A}$	
n	$\quad\mid \mathcal{B}$	
	$\mathcal{A} \to \mathcal{B}$	$\to$I, m–n

Another Example

$$P \to Q, Q \to R \therefore P \to R$$

1	$P \to Q$	
2	$Q \to R$	
3	$\quad P$	
4	$\quad Q$	$\to$E, 1, 3
5	$\quad R$	$\to$E, 2, 4
6	$P \to R$	$\to$I, 3–5

Introducing Subproofs

- As we will see, lots of rules will involve discharging assumptions, not just $\rightarrow$I, and so it is important we be clear about it now
- When we introduce an additional assumption, we begin a **subproof**, a proof within a proof
  - When you start a subproof by making the extra assumption $\mathcal{A}$, you are essentially asking: what could we show, if we added the assumption of $\mathcal{A}$ to what we already have?
- As we have seen, we indicate that we are working within a subproof by adding an extra vertical line

Working within a Subproof

- Subproofs are triggered when you add an extra assumption to your argument
- While you are working in that subproof, you can appeal to that extra assumption in the course of proving things
- You can also appeal to everything you already proved before triggering the subproof

Closing a Subproof

- As we have seen, some rules (e.g. $\to$I) allow you to discharge an assumption
- When you do that, the subproof which was triggered when you made that extra assumption becomes **closed**
- Once a subproof is closed, you are no longer allowed to appeal to anything from within that proof

A Bad Inference

$$P \to Q, Q \to R \therefore R$$

1	$P \to Q$	
2	$Q \to R$	
3	$\quad P$	
4	$\quad Q$	$\to$E, 1, 3
5	$\quad R$	$\to$E, 2, 4
6	$P \to R$	$\to$I, 3–5
7	R	$\to$E, 6, 3

A Bad Inference

1	$P \to Q$	
2	$Q \to R$	
3	P	
4	Q	$\to$E, 1, 3
5	R	$\to$E, 2, 4
6	$P \to R$	$\to$I, 3–5
7	R	$\to$E, 6, 3

- At line 7, we appealed to line 3, even though line 3 is stuck within a closed subproof

Sub-sub-proofs

- Once we have triggered a subproof by making an additional assumption, there is nothing to stop you from making **another** additional assumption, and triggering a subproof to the subproof
- In fact, sometimes you really **need** to embed subproofs in this kind of way
- For example, you need to do this to show that 'A' implies '$B \to (C \to (A \land B))$'

An Example

$$A \therefore B \rightarrow (C \rightarrow (A \wedge B))$$

1	A	
2	B	
3	C	
4	$A \wedge B$	$\wedge$I, 1, 2
5	$C \rightarrow (A \wedge B)$	$\rightarrow$I, 3–4
6	$B \rightarrow (C \rightarrow (A \wedge B))$	$\rightarrow$I, 2–5

Another Bad Inference

$$A \therefore C \to (A \land B)$$

1	A	
2	$\quad B$	
3	$\quad\quad C$	
4	$\quad\quad A \land B$	$\land$I, 1, 2
5	$\quad C \to (A \land B)$	$\to$I, 3–4
6	$B \to (C \to (A \land B))$	$\to$I, 2–5
7	$C \to (A \land B)$	$\to$I, 3–4

Citing Subproofs

1	A	
2	$\quad B$	
3	$\quad\quad C$	
4	$\quad\quad A \wedge B$	$\wedge$I, 1, 2
5	$\quad C \rightarrow (A \wedge B)$	$\rightarrow$I, 3–4
6	$B \rightarrow (C \rightarrow (A \wedge B))$	$\rightarrow$I, 2–5
7	$C \rightarrow (A \wedge B)$	$\rightarrow$I, 3–4

- When you use a rule which cites an entire subproof (like $\rightarrow$I), you cannot cite a subproof which occurs **within** a closed subproof

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Two Conditionals

- $\mathcal{A} \leftrightarrow \mathcal{B}$ can be thought of as the conjunction of two conditionals:
  - $\mathcal{A} \to \mathcal{B} \wedge \mathcal{B} \to \mathcal{A}$

- So if we wanted to prove $\mathcal{A} \leftrightarrow \mathcal{B}$, we (in effect) need to prove $\mathcal{A} \to \mathcal{B}$ and $\mathcal{B} \to \mathcal{A}$

Biconditional Introduction

l	$\quad\mid\;\mathcal{A}$	
m	$\quad\mid\;\mathcal{B}$	
n	$\quad\mid\;\mathcal{B}$	
o	$\quad\mid\;\mathcal{A}$	
	$\mathcal{A} \leftrightarrow \mathcal{B}$	$\leftrightarrow$I, l–m, n–o

Biconditional Elimination

m	$\mathcal{A} \leftrightarrow \mathcal{B}$	
n	$\mathcal{A}$	
	$\mathcal{B}$	$\leftrightarrow$E, m, n

m	$\mathcal{A} \leftrightarrow \mathcal{B}$	
n	$\mathcal{B}$	
	$\mathcal{A}$	$\leftrightarrow$E, m, n

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That, Or Anything Else

- Sharon studies archaeology. Therefore, either Sharon studies archaeology, or Sharon studies physics
- That argument is valid
  - A disjunction is true so long as one (or both) of the disjuncts is true; so if we assume that Sharon studies archaeology, we can infer that Sharon studies archaeology or studies physics
- In general, you can always infer $\mathcal{A} \vee \mathcal{B}$ from $\mathcal{A}$

Disjunction Introduction

m	$\mathcal{A}$	
n	$\mathcal{A} \vee \mathcal{B}$	$\vee$I, m

m	$\mathcal{A}$	
n	$\mathcal{B} \vee \mathcal{A}$	$\vee$I, m

What can You Infer from a Disjunction?

- Suppose you had proved $\mathcal{A} \vee \mathcal{B}$. What could you infer from that?
- Well, you know that either $\mathcal{A}$ is true, or $\mathcal{B}$ is true
- So if you could find some sentence, $\mathcal{C}$, which is implied by $\mathcal{A}$ and by $\mathcal{B}$, then you could infer that from the disjunction
- **Example:**
  - Either Sharon studies archaeology, or she studies physics. Either way, Sharon studies *something*. So Sharon studies something

Disjunction Elimination

i	$\mathcal{A} \vee \mathcal{B}$		
j		$\mathcal{A}$	
k		$\mathcal{C}$	
l		$\mathcal{B}$	
m		$\mathcal{C}$	
	$\mathcal{C}$		$\vee$E, i, j–k, l–m

An Example

$$(P \wedge Q) \vee (P \wedge R) \therefore P$$

1	$(P \wedge Q) \vee (P \wedge R)$	
2	$\quad P \wedge Q$	
3	$\quad P$	$\wedge$E, 2
4	$\quad P \wedge R$	
5	$\quad P$	$\wedge$E, 4
6	P	$\vee$E, 1, 2–3, 4–6

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Negation, via Contradiction

- There is one connective left to give some rules for: negation, '¬'
- However, it will be useful to approach negation via a new symbol: ⊥
- This symbol has gone by different of names — *falsum*, bottom — but we will call it the **contradiction** symbol
  - When we use the ⊥ symbol, you can think of it as the exclamation: But that is a contradiction!
- This symbol is governed by introduction and elimination rules, just like the other logical symbols

Contradiction Introduction

m	$\mathcal{A}$	
n	$\neg\mathcal{A}$	
	$\bot$	$\bot$I, m, n

Contradiction Elimination

m	$\bot$	
	$\mathcal{A}$	$\bot$E, m

Explosion

- The Contradiction Elimination rule relies on the idea that contradictions entail **everything**
- The old Latin name for this rule is *ex falso quodlibet*; it is commonly known today as Explosion
- That idea might seem shocking, but it is built into our definitions of validity

Explosion and the Intuitive Definition of Validity

- $\mathcal{A} \therefore \mathcal{B}$ is valid iff there is no world where $\mathcal{A}$ is true and $\mathcal{B}$ is false
- If $\mathcal{A}$ is a contradiction, then there is no world where $\mathcal{A}$ is true
- So there is no world where $\mathcal{A}$ is true **and** $\mathcal{B}$ is false
- So if $\mathcal{A}$ is a contradiction, then $\mathcal{A} \therefore \mathcal{B}$ is valid, *no matter what $\mathcal{B}$ is!*

Explosion and Tautological Entailment

- $\mathcal{A} \vDash \mathcal{B}$ iff there is no valuation on which $\mathcal{A}$ is true and $\mathcal{B}$ is false
- If $\mathcal{A}$ is a contradiction, then there is no valuation on which $\mathcal{A}$ is true
- So there is no valuation on which $\mathcal{A}$ is true **and** $\mathcal{B}$ is false
- So if $\mathcal{A}$ is a contradiction, then $\mathcal{A} \vDash \mathcal{B}$, *no matter what* $\mathcal{B}$ *is!*

Learning to Love Explosion

- Even though Explosion is built into our definitions of validity, some people think it looks wrong, and have constructed logical systems (known as *relevance logics*) which don't include it
- These systems are interesting, but very tricky, and we will not try to deal with them here
- And actually, there is a good case to make *in favour* of Explosion
  - **Question:** Why is it so bad to believe contradictions?
  - **Answer:** Because contradictions entail *everything*, and so if you believed a contradiction, you would have to believe every other sentence too!

The Negation Rules

- We do not need a new Negation Elimination rule: $\bot$I works as our Negation Elimination

m	$\mathcal{A}$	
n	$\neg\mathcal{A}$	
	$\bot$	$\bot$I, m, n

- However, we do still need a Negation Introduction rule
- The idea behind the rule is simple: if I manage to show that $\mathcal{A}$ leads to a contradiction, then I should be able to infer $\neg\mathcal{A}$

Negation Introduction

m	$\quad \mathcal{A}$ (assumption)	
n	$\quad \bot$	
	$\neg\mathcal{A}$	$\neg$I, m–n

Another Rule for Negation

- We also want to add one last rule governing negation
- Consider the following argument:
  - Either it is raining outside, or it is not
  - Suppose that it is raining; in that case, Simon will take an umbrella (to stay dry)
  - Suppose that it is not raining; in that case, Simon will take an umbrella (to avoid getting burnt)
  - So either way, Simon will take an umbrella
- The idea behind this argument is that a sentence is either true or false, and so if we can derive something from the supposition that it is true, and also from the supposition that it is false, then we are safe to infer it once and for all

Tertium Non Datur

i	$\quad \mathcal{A}$	
j	$\quad \mathcal{B}$	
k	$\quad \neg\mathcal{A}$	
l	$\quad \mathcal{B}$	
	$\mathcal{B}$	TND, i–j, k–l

Natural Deduction for TFL

Proof Strategies

Work backwards from what you want!

- Figure out what the *main logical connective* in your conclusion is
- That will give you one sensible suggestion for what the *second-to-last* line of your proof should be: it should set you up in a position to apply the appopriate introduction rule
- Now take the second-to-last line as your goal, and re-apply this method
  - Suppose that the conclusion is a conditional, $\mathcal{A} \rightarrow \mathcal{B}$
  - One obvious way of getting that conclusion is by applying $\rightarrow$I
  - This requires a subproof in which you assume $\mathcal{A}$, and then somehow infer $\mathcal{B}$
  - Now ask, How might I make that subproof?

Proof Strategies

Work forwards from what you have!

- When starting a proof, look at your premises, and ask what elimination rules you can apply to them
- Once you have derived some more sentences from your premises, look at what elimination rules you can apply to these new sentences
  - One very handy tip: if you have a premise whose main logical connective is conjunction, then it is often a good idea to apply conjunction elimination to it straight away

Proof Strategies

Don't be scared of making new assumptions when you need to!

- If you can't find any way of getting anything out of the premises you have, ask whether you could make any progress by adding a new assumption and triggering a subproof
- This can be very useful, but you always need to be careful when introducing new assumptions
  - Somehow, you will need to find a way of discharging the assumption which triggered the subproof, to return to the main proof!

Proof Strategies

If you get stuck, try using Tertium Non Datur

- Sometimes you have to use TND to make a proof work
- But that isn't always clear in advance: it is sometimes very surprising what you can use TND to prove
- So if you can't think of anythinge else to do, give TND a try

Proof Strategies

STICK WITH IT!!!

- Don't give up too quickly
- If one approach doesn't work, try another
- Slowly but surely, you will get a good handle on how to put proofs together!

Exercises

Give a proof for each of the following arguments

1. $A \to (B \to C) \therefore (A \wedge B) \to C$
2. $K \wedge L \therefore K \leftrightarrow L$
3. $(C \wedge D) \vee E \therefore E \vee D$
4. $J \to \neg J \therefore \neg J$

Exercises

Give a proof for each of the following arguments

5. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$
6. $\neg F \rightarrow G, F \rightarrow H \therefore G \vee H$
7. $(Z \wedge K) \vee (K \wedge M), K \rightarrow D \therefore D$
8. $P \wedge (Q \vee R), P \rightarrow \neg R \therefore Q \vee E$

Exercises

Give a proof for each of the following arguments

9. $S \leftrightarrow T \therefore S \leftrightarrow (T \vee S)$
10. $\neg(P \to Q) \therefore \neg Q$
11. $\neg(P \to Q) \therefore P$