Theories 3

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1 Euclidean Geometry

1 Given any two points P and Q, exactly one line can be drawn which passes through P and Q

2 Any line segment can be indefinitely extended

3 A circle can be drawn with any centre and any radius

4 All right angles are congruent to each other

5′ For every line l and for every point P that does not line on l, there is exactly one line m that can be drawn through P that is parallel to l

2 Hyperbolic Geometry

¬5 There exists a line l and point P not on l such that at least two distinct lines parallel to l pass through P

- For every line l and every point P not on l there are at least two distinct lines parallel to l which pass through P
- For any triangle ABC, the sum of the interior angles of ABC is strictly less than 180 degrees
- There are no rectangles
- All similar triangles are congruent (i.e. there are no triangles of the same shape but different sizes)

3 Relative Consistency

- Θ_2 is consistent relative to Θ_1 iff (if Θ_1 is consistent then Θ_2 is consistent)
- We show a theory Θ_2 is consistent relative to a theory Θ_1 in two steps
- First, we give an interpretation of the non-logical primitives of Θ_2 in the language of Θ_1

- Second we show that so interpreted, the sentences of Θ_2 are all theorems of Θ_1
- If Θ_1 is consistent it follows that Θ_2 is consistent when understood in this new way
- But (syntactic) consistency pays no attention to meaning, and so if Θ_2 is consistent on this understanding of its primitives, then it is consistent on every understanding of its primitives
- So, we thereby show that if Θ_1 is consistent then Θ_2 is consistent

4 Implicit Definition

- It is provable that if our deductive system is complete, any consistent set of sentences is satisfiable
- Γ is consistent iff there is no ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$
- Γ is satisfiable iff there is no ϕ such that $\Gamma \vDash \phi$ and $\Gamma \vDash \neg\phi$
- In other words, Γ is satisfiable iff some interpretation makes all of the sentences of Γ true; we call such an interpretation a *model* of Γ
- Assume that our deductive system is complete and a set of sentences Γ is consistent

  Now suppose that Γ is unsatisfiable

  In that case there is some ϕ such that $\Gamma \vDash \phi$ and $\Gamma \vDash \neg\phi$

  By completeness, there is some ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$

  So Γ is inconsistent

  Contradiction!

  Hence Γ is satisfiable