# Theories 2

Rob Trueman
rt295@cam.ac.uk

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## 1 A formal definition of axiom

- '$\Sigma$' will be used for axiom sets and '$\alpha$' for individual axioms (i.e. the members of axiom sets)
- An *axiom set* $\Sigma$ for a theory $\Theta$ is a finitely specifiable set of wffs from which every theorem of $\Theta$ can be deduced
- A set of wffs $\Gamma$ is *finitely specifiable* iff we can lay down explicit rules for determining of any given wff in a finite period of time whether or not it is in $\Gamma$
- One way I could finitely specify an axiom set $\Sigma$ is just to list a finite number of axioms and say that a sentence is a member of $\Sigma$ iff it appears on that list. Another is to use schemata
- A schema is a *shape* or *form* of a sentence. They have *instances*, which are sentences of the same shape
- We can stipulate that every instance of a given schema is to be an axiom

## 2 Peano Arithmetic

(1) $\forall x(0 \neq Sx)$

(2) $\forall x \forall y(Sx = Sy \supset x = y)$

(3) $\forall x(x + 0 = x)$

(4) $\forall x \forall y(x + Sy = S(x + y))$

(5) $\forall x(x \times 0 = 0)$

(6) $\forall x \forall y(x \times Sy = (x \times y) + x)$

(I) $((\phi(0) \wedge \forall x(\phi(x) \supset \phi(Sx))) \supset \forall x \phi(x))$

## 3 Two properties of theories

- A theory $\Theta$ is consistent if there is no $\phi$ such that $\Theta$ entails $\phi$ and $\neg\phi$
  - Syntactically: $\Theta$ is consistent iff there is no $\phi$ such that $\Theta \vdash \phi$ and $\Theta \vdash \neg\phi$
  - Semantically: $\Theta$ is consistent iff there is no $\phi$ such that $\Theta \vDash \phi$ and $\Theta \vDash \neg\phi$
- Typically people mean syntactic consistency by 'consistency'. Semantic consistency is often called 'satisfiability'
- A theory $\Theta$ is negation complete iff for every $\phi$, either $\Theta$ entails $\phi$ or $\Theta$ entails $\neg\phi$ (or both)
  - Syntactically: $\Theta$ is negation complete iff for every $\phi$, $\Theta \vdash \phi$ or $\Theta \vdash \neg\phi$
  - Semantically: $\Theta$ is negation complete iff for every $\phi$, $\Theta \vDash \phi$ or $\Theta \vDash \neg\phi$
- Typically people intend the syntactic understanding of 'negation completeness'
- Gödel proved that PA cannot be both consistent and negation complete!