Theories 1

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1 The definition of a theory

- A set of sentences is deductively closed iff every sentence that can be deduced from that set is itself a member of that set
- A set A is the deductive closure of a set of sentences B iff every sentence which can be deduced from B is a member of A, and every member of A can be deduced from B
- A theory is a deductively closed set of sentences (its members are its theorems)

2 Syntax

- A deductive system (or proof calculus) is a system of rules to determine which sentences can be *deduced* from other(s)
- Typically, the sign 'Θ' will stand for theories, '$\mathcal{L}$' for formal languages, 'Δ' for deductive systems, 'ϕ' and 'ψ' for wffs, and 'Γ' for sets of wffs
- The relation '$\vdash$' (the single turnstile) stands for deducibility. Remember, deducibility is relative to deductive systems: one sentence might be deducible from another in some systems and not in some other systems
- '$\phi \vdash \psi$' means that ψ can be deduced from ϕ
- '$\Gamma \vdash \phi$' means that the formula ϕ can be deduced from the formulae in Γ
- '$\vdash \phi$' means that ϕ can be deduced from nothing, i.e. ϕ is a logical truth of Δ
- '$\Theta \vdash \phi$' means that ϕ is a theorem of Θ
- Θ is a formal theory iff
  - $\Theta \subseteq \mathcal{L}$ (i.e. every sentence in Θ is a sentence of a formal language $\mathcal{L}$)
  - $\phi \in \Theta$ iff $\Theta \vdash \phi$ (i.e. Θ is deductively closed)

3 Semantics

- ψ is a logical consequence of ϕ iff every interpretation that makes ϕ true makes ψ true
- Typically I will use '$\mathcal{I}$' to stand for interpretations
- The relation '$\vDash$' (the double turnstile) stands for logical consequence. It can be used to relate interpretations to formulae and formulae to formulae
- $\mathcal{I} \vDash \phi$ means that ϕ is true under interpretation I. In other words: $\mathcal{I}$ *satisfies* ϕ
- '$\mathcal{I} \vDash \Gamma$' means that $\mathcal{I}$ satisfies every member of Γ
- '$\phi \vDash \psi$' means that every interpretation that satisfies ϕ satisfies ψ
- '$\Gamma \vDash \phi$' means that every interpretation which satisfies every member of Γ satisfies ϕ
- '$\vDash \phi$' means that every interpretation satisfies ϕ

4 Deducibility vs consequence

- A deductive system is *sound* iff (if $\Gamma \vdash \phi$ then $\Gamma \vDash \phi$)
- A deductive system is *complete* iff (if $\Gamma \vDash \phi$ then $\Gamma \vdash \phi$)