The Foundations of Mathematics
Lecture Nine

Neo-Fregean Logicism

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Neo-Fregean Logicism

Frege's Logicism

- In Lecture 1, we looked at Frege's logicism
- Frege originally considered introducing the concept *Number* with Hume's Principle, but rejected it because of the Julius Caesar Problem
- Instead, Frege defined numbers as classes, but his class theory turned out to be inconsistent
- That was the unhappy end to Frege's logicist project

Gottlob Frege

Hume's Principle

(HP) The number of Fs = the number of Gs iff F and G are equinumerous

$$NxFx = NxGx \equiv F \sim G$$

- There are two key points to note about (HP):
  (1) We can define what we mean by 'equinumerosity' without mentioning numbers in any way
  (2) (HP) entails all of arithmetic, if we are using second-order logic
- These two facts together make it very tempting to treat (HP) as some sort of definition of 'the number of...' (or '$Nx...x...$')

The Julius Caesar Problem

- (HP) tells us how to figure out whether a sentence of the form '$NxFx = NxGx$' is true:

  it's true just when the corresponding sentence of the form '$F \sim G$' is true

- But what about identity claims which are not of that form?

  (J) Julius Caesar $= Nx(x \neq x)$

- (HP) has now way to tell us whether or not (J) is true, i.e. whether or not Julius Caesar is the number 0
- But clearly we all know that (J) is false
- So there is more to our concept of *Number* than is contained in (HP)

Introducing the Neo-Fregeans

- Frege thought that the Julius Caesar Problem showed that we could not treat (HP) as any kind of definition of 'the number of...'
- Crispin Wright and Bob Hale, the neo-Fregeans, think that Frege over-reacted
- They think that we can solve the Julius Caesar Problem, and use (HP) as an implicit definition of 'the number of...'

Crispin Wright and Bob Hale

A Solution to Benacerraf's Dilemma

- Moreover, the neo-Fregeans argue that thinking of (HP) as an implicit definition of 'the number of...' will allow us to solve the two difficult problems posed by Benacerraf:
  (1) How can we know about numbers?
  (2) How can we even refer to numbers in the first place?
- If we are allowed to just stipulate (HP) as an implicit definition, then there is no mystery about how we can understand (HP), or know it to be true
- We then use (HP) to convert our unproblematic talk and knowledge about equinumerosity between properties into talk and knowledge about identity between numbers

Neo-Fregean Logicism

Disclaimer

- In this part of the lecture, I will present the original neo-Fregean solution to the Julius Caesar Problem
  - See Wright's *Frege's Conception of Numbers as Objects*, pp. 107–17
- But first a disclaimer: neo-Fregeanism is an ever-evolving doctrine, and this is no longer the official neo-Fregean solution
- But in my opinion, the now official solution is not as good as the original!
  - You can find it in Hale and Wright's 'To bury Caesar...' in their *The Reason's Proper Study*

The Guiding Idea

(HP) The number of Fs = the number of Gs iff F and G are equinumerous

- The guiding idea behind the neo-Fregean solution to the JCP is that (HP) is not just any old true principle about numbers; rather, when we take it as a definition, (HP) gives us insight into the nature of numbers
- So of course $Nx(x \neq x)$ isn't Julius Caesar:
  - Everyone agrees that (HP) would never tell you that $Nx(x \neq x)$ is Julius Caesar
  - So if (HP) tells us all about the true nature of the numbers, then $Nx(x \neq x)$ cannot be Julius Caesar!

Canonical Criteria of Identity for Numbers

(HP) The number of Fs = the number of Gs iff F and G are equinumerous

- (HP) supplies us with a criterion of identity for numbers
- According to the neo-Fregeans, if we stipulate (HP) as an implicit definition of 'the number of...', then we are treating this criteria of identity as somehow **canonical**
- (HP) doesn't just give us one way of deciding whether numbers are identical, it gives us **the way** that we are meant to use, the official canonical way

Canonical Criteria of Identity for People

- Julius Caesar = Augustus Caesar
- This is a sentence identifying one person with another person (falsely, in this case)
- There is a huge question about what the criteria of identity for people are, but we can all surely agree that the canonical criteria are nothing to do with equinumerosity!

Solving the Julius Caesar Problem

- According to the neo-Fregeans, this difference in canonical criteria of identity means that identity claims between numbers have a **different sort of content** from identity claims between people
- This in turn is meant to show that people cannot be numbers
  - If identity claims for numbers have a different sort of content from identity claims for people, then that must surely imply that numbers and people are different sorts of thing!

Neo-Fregean Logicism

Objecting to the Neo-Fregean Solution

- Lots of philosophers have presented lots of objections to the neo-Fregean solution to the JCP
- I think the most important objection is this:
  - In the process of "solving" the JCP, the neo-Fregeans take it for granted that '$NxFx = NxGx$' appears in (HP) as a real identity sentence
  - But there is a way of understanding the JCP as attacking that very assumption
  - On this way of looking at the JCP, the point is that despite appearances to the contrary, '$NxFx = NxGx$' isn't really an identity claim
  - Clearly, then, the neo-Fregeans have done nothing to answer this way of thinking about the JCP!

A Dilemma for Neo-Fregeanism

- What I am about to present is an edited down version of the argument in my 'A Dilemma for Neo-Fregeanism' (2014 in *Philosophia Mathematica*)
- I regard that argument as a version of the JCP, but I should warn now that that problem won't appear for a while, and when it does it is really just an illustration of a much deeper lying issue
- I should also mention that I am not at all sure that this is what Frege had in mind when he first presented the problem

The Structure of the Dilemma

(HP) The number of Fs = the number of Gs iff F and G are equinumerous

- It is clear that the neo-Fregeans need their "stipulation" of (HP) to do **two** things
- First, it must fix a meaning for 'the number of...'
  - Otherwise, (HP) would fail as an implicit **definition**
- Second, it must guarantee that (HP) as a whole is true
  - Otherwise, the fact that we can deduce all of arithmetic from (HP) would give us no reason to believe that arithmetic is true

The Structure of the Dilemma

- I will distinguish between two different senses in which the neo-Fregeans might be “stipulating” (HP)
- One of them fixes a meaning for ‘the number of...’
- One of them guarantees that (HP) is true
- Neither does both

Neo-Fregean Logicism

Subsentential Stipulation

- On the first way of understanding what the neo-Fregeans are doing when they "stipulate" (HP), they are issuing the following resolution:
  - 'The number of...' is to refer to whatever function it needs to refer to for (HP) to be true
- Since this focusses on a subsentential part of (HP) — 'the number of...' — I will call it the **subsentential stipulation**

Subsentential Stipulation: The Good News

- I think we should all agree that the subsentential stipulation **does** fix a meaning for 'the number of...'
- It gives us a way of deciding whether a given function is the function referred to by 'the number of...', and that seems like enough
- Indeed, this stipulation seems to work in **exactly the same way** as this less exotic stipulation:
  - 'Jack the Ripper' is to refer to whomever it needs to refer to for 'Jack the Ripper committed the Whitechapel murders' to be true

Subsentential Stipulation: The Bad News

- But as well as fixing a meaning for 'the number of...', the stipulation of (HP) is mean to guarantee that (HP) is true
- This is something that the subsentential stipulation does not do!
- It is helpful again to compare the subsentential stipulation to our definition of 'Jack the Ripper':
  - 'Jack the Ripper' is to refer to whomever it needs to refer to for 'Jack the Ripper committed the Whitechapel murders' to be true
  - × Jack the Ripper committed the Whitechapel murders
  - ✓ **If** someone committed the Whitechapel murders, **then** Jack the Ripper did

Subsentential Stipulation: The Bad News

- The subsentential sitpulation does not guarantee that (HP) is true: for all it tells us, it might be that **no** function behaves in the way that (HP) requires
- The most that the subsentential stipulation can give us is the following conditional:

If there is some f s.t. $f(F) = f(G)$ iff $F \sim G$,
then the number of Fs = the number of Gs iff $F \sim G$

The Neo-Fregeans Already Knew This!

> *It goes with the [subsentential] model that it must be at least initially intelligible that a principle proposed in this spirit fails to hit off reference to anything. It cannot just be a given that reference is secured, even if it is.*
>
> *Hale and Wright (2009) 'The meta-ontology of abstraction', p. 206*

Neo-Fregean Logicism

Sentential Stipulation

- Here is another way of understanding what the neo-Fregean stipulation of (HP):
  - Every instance of '$NxFx = NxGx$' is to have the same truth-value as the corresponding instance of '$F \sim G$

    (eg: 'The number of knives = the number of forks' is to have the same truth-value as 'the knives and the forks are equinumerous')

- Since this focusses on a the whole sentence (HP), I will call it the **sentential stipulation**

The Prospects for Sentential Stipulation

- The good news is that the sentential stipulation surely does guarantee that (HP) is true
  - (HP) is a biconditional, and so is true so long as both sides share a truth-value, which is just what the sentential stipulation stipulates!
- The bad news is that it is not at all clear how the sentential stipulation can do the second thing it needs to do: fix a meaning for 'the number of...'

The Austere Reading

- The sentential stipulation tells us nothing about the inner workings of a sentence of this form:

  (1) The number of Fs = the number of Gs

- It just tells us that an instance of (1) is to share its truth-value with the corresponding instance of:

  (2) The Fs and the Gs are equinumerous

The Austere Reading

(1) The number of Fs = the number of Gs

(2) The Fs and the Gs are equinumerous

- Surely, then, we have no right to read any complexity into (1) beyond the variables 'F' and 'G'?
- (1) is an unstructured open sentence with 'F' and 'G' as variables, nothing more than a handy abbreviation for (2)

  (1′) $R(F, G)$

- Call this way of reading (1) the **austere reading**
- The neo-Fregeans require the **robust** reading of (1), on which (1) has exactly the form it appears to have

The Neo-Fregean Justification of the Robust Reading

> *What a recipient of (HP) immediately learns is that whatever suffices for the truth of a statement of one-to-one correspondence is equally sufficient for the truth of the corresponding statement of number-identity. However, she also understands that she is to take the surface syntax of number-identity at face value. She already possesses the general concept of identity, and so is able to recognise that the expressions flanking the identity sign must be singular terms.*
>
> *(Hale and Wright, 'Benacerraf's dilemma revisited', EJP 2002, pp. 117–8)*

Against the Robust Reading

(1) The number of Fs = the number of Gs

(2) The Fs and the Gs are equinumerous

- The neo-Fregean justify the robust reading of (1) on the grounds that we **already** understand that '=' is the identity sign, and so 'the number of Fs' and 'the number of Gs' must be singular terms
- But I do not think that the neo-Fregeans are free to **first** stipulate that every instance of (1) is to share a truth-value with the corresponding instance of (2), **and then** insist that '=' appears in (1) as the identity sign
- To see why, it will be useful to start with a simpler, toy case

A Toy Case

- Imagine I simply stipulated that 'F(Socrates)' is to be a true sentence
- Could I then insist that 'Socrates' appears in 'F(Socrates)' as a term referring to Socrates?
- Hale and Wright present one obvious problem with trying to do so:
  - If 'Socrates' appears in 'F(Socrates)' as a term, then we should be able to intelligibly substitute other terms for it
  - But we can't: we have no idea, for example, what 'F(Plato)' might mean
- This is a good point, but I think it is really just a symptom of a deeper lying problem

A Toy Case

- Now imagine that I introduced 'G(Socrates)' by stipulating that 'G' is to be true of an object iff that object committed suicide
- In this case, the truth-value of 'G(Socrates)' is determined in part by the reference of 'Socrates'
  - 'G(Socrates)' is true iff the referent of 'Socrates' committed suicide
- And clearly, in this case, we can understand 'G(Plato)'
  - All we need to do to get from the truth-conditions of 'G(Socrates)' to the truth-conditions of 'G(Plato)' is swap out a reference to Socrates for a reference to Plato

A Toy Case

- Things were wholly different in the case of ‘F(Socrates)’:
  - No reference to Socrates played any role in determining the truth-value of ‘F(Socrates)’: we just outright stipulated that this sentence is true
- And that is why we cannot understand ‘F(Plato)’
  - When it comes to determining the truth-value of ‘F(Socrates)’, ‘Socrates’ is not a relevant unit ready to be replaced

A Toy Case

- Moreover, this is a deeper point than Hale and Wright's:
  - The link between reference and truth-value is non-opitional
  - Part of what it is for ‘Socrates’ to refer to Socrates is for that term to play a role in determining the truth-values of the sentences in which it appears
  - So if the apparent reference of ‘Socrates’ plays no role in determining the truth-value of ‘F(Socrates)’, then that just means that ‘Socrates’ isn’t really a referring term in that sentence at all!

The Lesson from the Toy Case

- When we fix a truth-value for a sentence, the way in which we do so settles what, if any, roles the parts of the sentence play in determining that truth-value
- So if we want ‘Socrates’ to appear as a name of Socrates in a given sentence, we are thereby restricted in the ways in which we can fix the truth-value of that sentence
- We must do so in a way that assigns the appropriate role to ‘Socrates’
  - The fact that ‘Socrates’ refers to Socrates must have a knock-on effect on the truth-value of the whole sentence
- We conformed to this restriction with ‘G(Socrates)’, but not with ‘F(Socrates)’

The Neo-Fregean Case

(1) The number of Fs = the number of Gs

(2) The Fs and the Gs are equinumerous

- Can neo-Fregeans stipulate that instances of (1) are to have the same truth-value as instances of (2), **and then** insist that '=' appears in (1) as the identity sign?
- Here is one obvious problem with trying to do that:
  - If '=' appears in (1) as a predicate, then 'The number of Fs = ...' appears in (1) as a complex predicate
  - We should be able to substitute other predicates for this complex predicate
  - But if we try to substitute 'Julius Caesar = ...' for this predicate, we get 'Julius Caesar = the number of Gs', and we have no idea how to understand that

The Neo-Fregean Case

- That was obviously a version of the JCP, but as in the toy case, I think that this is just a symptom of a deeper problem
- When we sententially stipulate (HP), we are fixing the truth-values of the instances of (1)

  (1) The number of Fs = the number of Gs

- They are to have the same truth-values as the corresponding instances of (2)

  (2) The Fs and the Gs are equinumerous

The Neo-Fregean Case

(1) The number of Fs = the number of Gs

(2) The Fs and the Gs are equinumerous

- But when we fix the truth-values of the instances of (1) in this way, the supposed fact that '=' appears as the identity sign plays no role in determining those truth-values
- By the same token, 'the number of Fs = ...' plays no such role, which is why we cannot understand 'Julius Caesar = the number of Gs'
  - When it comes to determining the truth-values of instances of (1), 'the number of Fs = ...' is not a relevant unit ready to be replaced

The Neo-Fregean Case

- Why is this a deeper problem than the initial appearance of the JCP?
- At least part of what it is for '=' to play the role of the identity sign is for it to play a particular role in determining the truth-values of the sentences in which it appears
- So to admit that '=' plays no such role in

  (1) The number of Fs = the number of Gs

  is to admit that '=' does not really appear as the identity sign

The Neo-Fregean Case

- But in that case, the neo-Fregeans are just stuck with the austere reading of (1)

  (1) The number of Fs $=$ the number of Gs

- (1) is not really an identity claim, as its surface form suggests, but just a handy abbreviation for

  (2) The Fs and the Gs are equinumerous

- Thus the sentential stipulation fails to fix a meaning for 'the number of...'!

Conclusion

- There are two ways in which we could "stipulate" (HP)
- **Subsentential Stipulation**
  - ✓ Fixes a meaning for 'the number of...'
  - × Doesn't guarantee that (HP) as a whole is true
- **Sentential Stipulation**
  - ✓ Guarantees that (HP) as a whole is true
  - × Doesn't fix a meaning for 'the number of...'

With Thanks to Mary Leng

I would like to thank Mary Leng, who let me borrow freely from her notes in preparation for these lectures