# The Foundations of Mathematics
## Lecture Seven

# Field's Programme

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# Field's Programme

# Benacerraf's Dilemma: Semantics

- We are pushed towards platonism by our best semantics for mathematics
  - A good semantics for mathematics will treat existential claims in maths in the same way that they are treated in other parts of our language
  - So 'There is a prime number greater than 17' says that a certain mathematical object **exists**
  - So since 'There is a prime number greater than 17' is true, mathematical objects exist

# Benacerraf's Dilemma: Epistemology

- We are pushed towards nominalism by our best epistemology for mathematics
  - If mathematical entities exist, then they are abstract
  - But it is a mystery how we could know anything about abstract objects
  - So if maths really is about mathematical entities, then it is a mystery how we can know anything about maths

# The Indispensability Argument

- The **Indispensability Argument** is an argument for platonism

  (1) We should believe our best scientific theories
  (2) Mathematics is indispensable to our best scientific theories
  (3) Our theories get confirmed or disconfirmed **as whole theories**

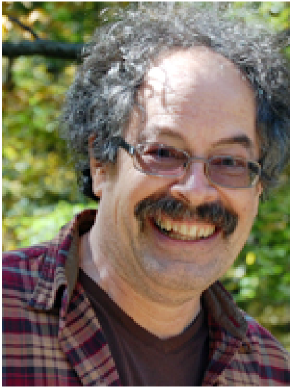
  $\therefore$ (4) We should believe that there are some mathematical objects

- Any nominalist will have to reject at least one of (1)–(3)

# Field's Programme

# Field's Fictionalism

- Field is a nominalist: he does not believe in mathematical objects
- But Field also thinks that our mathematical theories are ontologically committed to mathematical objects
  - Our mathematical theories can be true only if mathematical objects exist
- So Field concludes that our mathematical theories are false!
- Field's position is known as **fictionalism**



Hartry Field

# Field on Benacerraf

- The dilemma was meant to be this:
  - We are pulled towards platonism by our semantics...
  - ... but we are pulled towards nominalism by our epistemology
- Field entirely agrees with the idea that our epistemology pulls us towards nominalism
  - Field is a nominalist **because** he cannot see how we could know about (or refer to) abstract objects
- But he denies that our semantics pulls us towards platonism
  - Field agrees that we should read 'There is a prime number greater than 17' as a bona fide existential claim
  - So **if** 'There is a prime number greater than 17' is true, **then** numbers exist
  - But Field denies that it is true!

# Field on Dispensability

(1) We should believe our best scientific theories

(2) Mathematics is indispensable to our best scientific theories

(3) Our theories get confirmed or disconfirmed **as whole theories**

∴ (4) We should believe that there are some mathematical objects

- Field rejects premise (2)
- According to Field, mathematics is **not** indispensable from science
- Although scientists standardly use mathematics in their theories, you could formulate **nominalistic** versions of those theories which do not mention any numbers at all

# Why should we Trust Mathematics?

- Although Field thinks that we **could** present nominalistic versions of our scientific theories, he is happy to admit that the mathematised are much easier to work with
- This raises a **big** question for fictionalism:
  - If our mathematical theories are **false**, then why should we trust mathematised science?
- Field's answer: because mathematics is **conservative** over our nominalistic theories
  - $M$ is conservative over $N =_{df}$
    for any nominalistic sentence $\phi$, $M + N$ entails $\phi$ iff $N$ entails $\phi$

# What is the Point of Mathematics?

- According to Field, mathematics is a **useful** fiction
- Mathematised science is much easier to work with than nominalistic science
- But because our mathematical theories are conservative over nominalistic theories, we know that mathematised science will never mislead us

# Field's Programme

# Science without Numbers

- In his landmark book, *Science without Numbers*, Field starts to work through some of the details of his programme
- He tried to do three things:
  1. Provide a nominalistic version of Netwonian Gravitational Theory
  2. Provide bridge laws between this nominalistic theory and mathematics
  3. Prove that mathematics is conservative over this nominalistic theory

# Mathematatised Newtonian Gravitational Theory

- Ordinary Newtonian Theory makes free use of mathematics
- We represent individual spacetime co-ordinates with quadruples of real numbers
- We represent regions of spacetime with sets of quadruples of real numbers
- We then use mathematical relations and functions to express the gravitational theory
  - For example, 'the gravitational potential of $x$' expresses a function from quadruples of real numbers to real numbers

# Nominalistic Newtonian Gravitational Theory

- Field's nominalistic Newtonian theory gets rid of all these mathematical entities
- Rather than using quadruples of real numbers to represent spacetime points, the theory just quantifies over the spacetime points themselves (which it treats as real objects)
- Rather than using sets of quadruples of real numbers to represent spacetime regions, the theory just quantifies over the spacetime regions themselves (which it treats as real objects)

# Nominalistic Newtonian Gravitational Theory

- Rather than expressing the theory in terms of mathematical relations, it uses physical relations that hold between spacetime points and regions
  - For example, Field does not use 'the gravitational potential of $x$', which expresses a **mathematical** function
  - Instead, he uses 'the difference in gravitational potential between $x$ and $y$ is less than that between $z$ and $w$', which expresses a **physical** relation between spacetime points

# Nominalistic Newtonian Gravitational Theory

- It is generally accepted that Field's nominalistic Newtonian Gravitational Theory is **empirically equivalent** to ordinary, mathematised Newtonian Theory
- So on the face of it, Field has completed his first task:
  (1) Provide a nominalistic version of Netwonian Gravitational Theory
- Now we turn to his second task:
  (2) Provide bridge laws between this nominalistic theory and mathematics

# Representation Theorems

- Field needs to provide us with bridge laws that connect his nominalistic Newtonian theory with mathematics
  - He needs those bridge laws to explain how we can apply mathematics to the nominalistic theory
- For Field, these bridge laws take the form of **representation theorems**
- These representation theorems tell us that we can use certain mathematical entities to represent certain physical states
  - We obviously have to use **mathematics** to prove these theorems
  - But that is not a problem if mathematics really is conservative over the nominalised Newtonian theory (more on that shortly)

# Conservativeness

- We come now to Field's final task:
  (3) Prove that mathematics is conservative over this nominalistic theory
- Field offers a philosophical reason for accepting (3):
  - Even the **platonist** would agree that it would be odd to think that claims about **abstract mathematical entities** could possibly have any consequences for **concrete physical entities**

# Conservativeness

- But Field gives us more than that: he offers a **proof** of conservativeness
- We will come back to what exactly Field proved shortly
- First, we should note one odd fact about Field's proof:
  - In the course of offering this proof, Field helps himself to lots of mathematics, which is odd for a nominalist
  - Field's official line is that his argument is meant to convince his **opponents**, the platonists, that there is nothing wrong with his nominalism

# Field's Programme

# Spacetime Points

- There is a very clear sense in which Field simply replaces quadruples of real numbers with **spacetime points**
- But you might well wonder how much of an improvement that is, from a nominalist point of view
- On the face of it, spacetime points are not physical objects, and they are not things that we an interact with
- So how exactly are spacetime points any better than numbers?
- Indeed, lots of philosophers and physicists have been sceptical about the existence of spacetime points for broadly the same reasons that philosophers have been sceptical of numbers

# Spacetime Points

- Field's response to this objection is to insist that, according to modern physics, we **do** causally interact with spacetime points
- Physics has for a long time posited fields (e.g. electromagnetic fields) which are spread out throughout spacetime
- According to Field, when a physicist says that an electromagnetic field has a given strength at a given spacetime point, we should understand them as attributing an electromagnetic strength **to the spacetime point**
- So spacetime points have causal powers, which is how we can interact with them, and thus know about them

- It goes without saying that this is a controversial way of thinking about fields, but it is by no means implausible

# Other Scientific Theories

- Let's grant that Field has successfully nominalised Newtonian Gravitational Theory
- That is just one scientific theory (and an out of date one at that!)
- Field must nominalise **every** scientific theory we have
- Moreover, it is not at all clear that Field will be able to use his nominalisation of Newtonian Theory as a template
- In the Newtonian case, it was a simple matter of replacing quadruples of real numbers with spacetime points, but other theories (e.g. quantum mechanics) use very sophisticated, abstract mathematics
- It is not at all clear that we will be able to eliminate that kind of maths in favour of some physical counterpart

# Field's Programme

# Field versus Hilbert

Hartry Field

- There are clear similarities between Field's fictionalist programme and Hilbert's formalist programme
- Hilbert's programme was scuppered by Gödel's Incompleteness Theorems
- The Incompleteness Theorems pose a threat to Field's programme too

David Hilbert

# Field and Hilbert: The Similarities

- Field and Hilbert both thought that lots of modern mathematics is not true
  - In one regard, Hilbert was more extreme than Field, because Hilbert thought that ideal mathematics is **meaningless**, whereas Field just thinks that it is false
  - In another regard, Field is more extreme than Hilbert, because Hilbert thinks that finitary maths is fine, but Field thinks that **all** of our mathematical theories are false
- Both Field and Hilbert wanted to explain why the dodgy mathematics is still useful by saying that it is a conservative extension of something in good order
  - For Hilbert, ideal maths is conservative over finitary maths
  - For Field, all maths is conservative over nominalistic science

# Gödel's Incompleteness Theorems

- Hilbert's theory was destroyed by Gödel's Theorems
- Shaprio showed that those theorems also cause trouble for Field in his 'Conservativeness and Incompleteness'

# Modelling Robinson Arithmetic

- Consider a line of spacetime points in Field's nominalistic Newtonian Theory
- This line has all of the structure of the mathematical real number line
- Consequently, we can model the natural number line within that line
  - The natural number line just has less structure than the real number line
- Moreover, we can formulate versions of all of the axioms of Robinson Arithmetic which apply to this model of the natural number line

# Incompleteness and Conservativeness

- So if we assume that Field's nominalsitic theory ($N$) is consistent and effectively axiomatisable, then Gödel's Second Incompleteness Theorem kicks in:
  - $N \nvdash \mathrm{Con}_N$
- However, if we help ourselves to enough maths, we can prove that $N$ is consistent:
  - $M + N \vdash \mathrm{Con}_N$
- So our mathematics is not conservative over $N$!

# What did Field Prove?

- Earlier I told you that Field **proved** that mathematics is conservative over $N$
- What is going on?
- Conservativeness is defined in terms of **entailment**:
  - $M$ is conservative over $N =_{df}$
    for any nominalistic sentence $\phi$, $M + N$ entails $\phi$ iff $N$ entails $\phi$
- But as I explained in Lecture 5, there are two notions of entailment:
  - Syntactic deducibility: $\vdash$
  - Semantic logical consequence: $\models$

# What did Field Prove?

- So there are two different versions of conservativeness:
  - $M$ is **deductively** conservative over $N =_{df}$ for any nominalistic sentence $\phi$, $M + N \vdash \phi$ iff $N \vdash \phi$
  - $M$ is **semantically** conservative over $N =_{df}$ for any nominalistic sentence $\phi$, $M + N \vDash \phi$ iff $N \vDash \phi$
- What Field proved was that mathematics is **semantically** conservative over $N$
- What Shapiro proved (using Gödel's Theorems) was that mathematics is not **deductively** conservative over $N$

# But How Is That Possible?

- You might be wondering how mathematics could be semantically conservative but not syntactically conservative
- (Some versions of) Field's nominalistic Newtonian Theory was **second-order**
- And as we saw in Lecture 5, the standard deductive system of second-order logic is not complete relative to the standard semantics:
  - It is not the case that for every sentence $\phi$ and every set of sentences $\Gamma$: $\Gamma \vDash \phi \rightarrow \Gamma \vdash \phi$

# Field and Second-Order Logic

- It is not quite easy to say how bad this is for Field
- It certainly forces Field to introduce some elements into his system which do not look very **nominalistic**
- First off, second-order quantifiers are usually thought of as either quantifying over sets, or quantifying over properties
  - $\exists F\ Fa \Rightarrow$ there is some set $F$ such that $a \in F$
  - $\exists F\ Fa \Rightarrow$ there is some property $F$ such that $a$ instantiates $F$
- Neither of these ways of reading second-order logic look good for a nominalist
  - Can we causally interact with sets or properties?

# Field and Second-Order Logic

- Field tries to get out of this by reading second-order quantification as quantification over regions of spacetime points
  - $\exists F\ Fa \Rightarrow$ there is some region $F$ such that $a$ is in $F$
- However, this is a theorem of second-order logic:
  - $\exists x \phi x \rightarrow \exists F(Fx)$
- For Field, this says that if some spacetime point, $x$, is $\phi$, then there is some region $x$ is in
- But $\phi$ could be any predicate you like, even one which does not mention regions
- So it looks like pure logic is **introducing** commitment to regions, and Field thought logic shouldn't introduce **any** ontological commitments

# Field and Logical Consequence

- It also isn't clear whether Field can really help himself to the notion of **semantic** logical consequence
- Logicians usually define this notion in terms of **sets**, but Field doesn't really believe in mathematical entities
- Field tries to get around this problem by defining logical consequence in terms of **logical possibility**
  - $\phi \vDash \psi \ =_{df} \ \neg \diamond (\phi \wedge \neg \psi)$
- Field then suggests that we should just take this notion of logical possibility as primitive

# Field and Logical Consequence

- However, it really is not clear why a **nominalist** is allowed to take logical possibility as primitive
- We have no way of causally interacting with the merely "logically possible"
- So how can we know anything about logical possibility?
  - Remember, Field is distinguishing logical possibility from **consistency**, which is just a **syntactic** matter
- In particular, consider the claim: $\diamond$(there are infinitely many objects)
- How could Field ever know anything like that?

# For the Seminar

- Required reading:
  - Field, 'Realism and Anti-Realism about Mathematics', ch. 2 in his *Realism, Mathematics and Modality*
  - MacBride (1999) 'Listening to Fictions', *The British Journal for the Philosophy of Science*, vol. 50 pp. 431–55
- You can find links to both of these on the VLE

# For Next Week

- Next week we will be looking at structuralism
- Please read the following
  - *Shapiro* ch. 10
  - Benacerraf, 'What Numbers Could not Be' in *B&P*