The Foundations of Mathematics
Lecture Six

Benacerraf's Dilemma
&
The Indispensability Argument

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Benacerraf's Dilemma and the Indispensability Argument

The Special Properties of Mathematics

- In previous lectures, I have emphasised that mathematics at least **appears** to have some special properties:
  - Mathematical truths are **necessarily** true
  - Mathematical truths can be known **a priori**
  - Mathematical truths can be known with **certainty**
  - Mathematics deals with **infinities**
- By my lights, it is this last point about infinity that makes the philosophy of mathematics **so difficult**
- Again and again, we have seen philosophers grapple with infinity, and none of them seem to come out of it too well

The Contemporary Perspective

- However, issues surrounding infinity have taken a bit of a backseat in the contemporary philosophical discussions of mathematics
- These days, philosophers tend to be much more interested in questions about the **ontology** of mathematics
  - Do numbers exist?
  - If so, what sort of thing are they, and what sort of relations can we bear to them?
  - If not, then what is mathematics about?
- Of course, we have also been concerned with these questions all along, but now they take centre stage ...
- ... nonetheless, we will certainly see questions concerning infinity coming up again and again!

Platonism versus Nominalism

- **Platonism** (or "realism in ontology")
  - Numbers and other mathematical objects exist
  - They are non-physical **abstract objects**: they are outside of time and space, and have absolutely no causal powers
  - They exist **independently** of us
- **Nominalism**
  - There are no mathematical objects
  - Taken at face value, any theory which appears to describe mathematical objects is **cannot** be true: there are no mathematical objects!
  - The only way that these theories can be true is if there is some way of understanding them as not really talking about mathematical objects

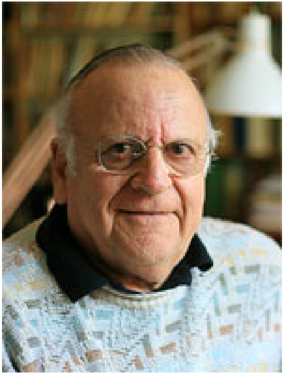
Platonism versus Nominalism

- Platonism and nominalism are clearly extreme positions
- It might be possible to find some sort of middle ground
- Nonetheless, most contemporary philosophers find themselves leaning more towards one of these extremes or the other
- The debate between platonism and nominalism is primarily driven by **Benacerraf's Dilemma**

Benacerraf's Dilemma and the Indispensability Argument

Paul Benacerraf

- Benacerraf introduced his dilemma in his 'Mathematical Truth' (reprinted in *B&P*)
- This dilemma became **the** problem to solve in the philosophy of mathematics
- Benacerraf also wrote another very influential paper, 'What Numbers Could Not Be', which we come back to in Lecture 8



Paul Benacerraf

The Dilemma in a Nutshell

- Our philosophy of mathematics needs to give us (at least) two things:
  (i) A **semantics** for mathematics
  (ii) An **epistemology** for mathematics

- A **semantics** will explain what it is for a mathematical sentence to be **true**
- According to Benacerraf, our attempts to develop such a semantics tends to drive us towards platonism

- An **epistemology** will explain how it is that we can **know** things in mathematics
- According to Benacerraf, our attempts to develop such a semantics tends to drive us towards nominalism

Truths Mathematical and Otherwise

- Consider the following two sentences:
  - (1) There is a city which is bigger than York
  - (2) There is a prime number which is greater than 17
- On the face of it, these two sentences seem to have the same **form**:
  - (3) There is an F which bears relation R to a
- And since they have the same form, it seems sensible to say that they should have the same kind of **truth-conditions**
- (1) is true just in case some object has a certain property and stands in a certain relation to a certain object
- So surely the same goes for (2)?

A Homogenous Semantics

A theory of truth for the language we speak, argue in, theorize in, mathematize in, etc., should [...] provide similar truth conditions for similar sentences. The truth conditions assigned to two sentences containing quantifiers should reflect in relevantly similar ways the contributions made by the quantifiers. Any departure from a theory thus homogeneous would have to be strongly motivated to be worth considering.

(Benacerraf, 'Mathematical Truth' in B&P p. 404)

From Semantics to Platonism

- Benacerraf is urging that we should understand the quantifiers in mathematical sentences in **exactly the same way** as we understand them in non-mathematical sentences
- We ordinarily understand the quantifier 'There is...' as expressing **existence**, so that is how we should understand it in maths
- Thus, if we accept our mathematical theories, which include sentences starting 'There is...', then we must accept the existence of mathematical objects
- We are thus pushed towards platonism

The Access Problem

- According to platonism, mathematical objects are **abstract** objects
- It is not **entirely** clear what this means, but its generally accepted to entail the following things:
  - Mathematical objects are not located anywhere in space
  - Mathematical objects are not located anywhere in time
  - Mathematical objects have no causal powers
- Numbers thus seem far removed from us ordinary humans living in space and time
- This has led to the **Access Problem**:
  - How exactly can we know anything about these abstract mathematical objects?

The Gettier Problem

- From Plato until the 1960s, it was generally agreed that knowledge was justified true belief
- But then Gettier published some famous examples where people have justified true belief, but seem to lack knowledge

> I go to see my doctor, who does some tests to see if I have a disease. He sends these tests off for analysis, and a few days later I get a letter from the doctor saying that I do not have the disease. I come to believe that I do not have the disease, and clearly have good justification for that belief. What is more, I really do not have the disease. **However:** my test results got muddled up with someone else's, and so what I actually received were their results. Surely I do not really **know** that I do not have the disease?

The Causal Theory of Knowledge

- Most philosophers responded by accepting that knowledge is not (or not **merely**) justified true belief
- For a brief while, it was popular to give a **Causal Theory of Knowledge**:
  - To know that p, your belief that p must be caused, in an appropriate way, by the fact that p
- This was the kind of theory that Benacerraf accepted
- And it is easy to see that this theory of knowledge would make it impossible to know mathematical truths:
  - Your belief that there is a prime number greater than 17 is not caused by the fact that there is a prime number greater than 17, because numbers have no causal powers!

Other Theories of Knowledge

- The Causal Theory of Knowledge didn't stay popular very long
- But that does not immediately dispense with the Access Problem!
- One way or another, we need some account of how our mathematical beliefs can **reliably reflect** how things are with numbers
- But it can be very difficult to see what kind of account we could possibly give: Numbers are abstract objects, completely removed from us!

  (See Field's discussion in his 1998 *Realism, Mathematics and Modality*, pp. 25–30)

From Epistemology to Nominalism

- To be clear: Benacerraf is not wanting to deny that we know mathematical truths
- He takes it for granted that we **do** have such knowledge
- The problem is that platonism seems to make that knowledge totally mysterious!
- It seems, then, that this gives us a good reason to reject platonism and move towards nominalism
- If we were nominalists, then we would deny that mathematical truths are really about abstract objects, and so would not have to deal with the platonist's Access Problem

Semantics versus Epistemology

- A good **semantics** for mathematics will deal with quantification in mathematical sentences in the same way that we deal with quantification in non-mathematical sentences
- This pushes us towards platonism: if there are mathematical truths beginning 'There is...', then mathematical objects exist

- A good **epistemology** for mathematics will make our knowledge of mathematical truths unmysterious
- This pushes us towards nominalism: if mathematical truths were about abstract objects, then we would be confronted with the Access Problem

Semantics versus Semantics

- In some ways, it is odd that Benacerraf set up his dilemma as a conflict between semantics and epistemology
- He **could** have set it up as a conflict **within semantics**
- If our ability to know about abstract mathematical objects would be mysterious, then surely our ability to **refer** to abstract mathematical objects would be mysterious to!
- We cannot interact with abstract objects, so how would we refer to them?

(This problem is particularly sharp if we accept a causal theory of reference, which Benacerraf did. What is more, causal theories of **reference** have proven much more popular than causal theories of **knowledge**.)

Semantics versus Semantics

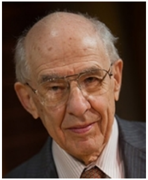
- Thus we are pulled in two directions in our semantic theorising:
  - We want our semantics for mathematics to look basically the same as our semantics for everything else
  - But we don't want our ability to refer to mathematical objects to be mysterious

Benacerraf's Dilemma and the Indispensability Argument

Quine and Putnam

- The **Indispensability Argument** is normally attributed to Quine and Putnam
- This argument has been almost as influential on the philosophy of maths as Benacerraf's Dilemma
- It is an argument for platonism, but more than that, it also provides an explanation of how we can know things about abstract mathematical objects

WVO Quine



Hilary Putnam

The Indispensability Argument

(1) We should believe our best scientific theories

(2) Mathematics is indispensable to our best scientific theories

(3) Our theories get confirmed or disconfirmed **as whole theories**

∴ (4) We should believe that there are some mathematical objects

Naturalism

(1) We should believe our best scientific theories

- This premise seems completely uncontroversial
- It is sometimes confused with **naturalism**:
  - Natural science is the ultimate arbiter of **all** our knowledge
- Naturalism is very popular (although I suspect it is deeply mistaken), but nonetheless it is important to see that it is not the same as Premise (1)
- (1) does not tell us that we should always look to science if we want to know something; it just tells us that we should believe in whatever science tells us

The Indispensability of Mathematics

(2) Mathematics is indispensable to our best scientific theories

- Consider Newton's Law of Gravitation:

  $F = G\frac{M_a M_b}{d^2}$

- In this law, we associate numbers with certain physical properties: we use numbers to describe the masses of *a* and *b*, to describe the distance between *a* and *b*, and to describe the force of attraction between them; the law then tells us how these numbers all relate to each other
- Moreover, more complex physical theories, like General Relativity and Quantum Mechanics, refer to even more exotic mathematical entities, like metric tensors and vectors in Hilbert Spaces

Confirmational Holism

(3) Our theories get confirmed or disconfirmed as whole theories

- This is known as **confirmational holism**, and it is one of Quine's most famous doctrines
- According to Quine, we cannot confirm or disconfirm **one sentence at a time**
- If we want to test a certain claim, we always have to make lots of **background assumptions**
- So what we are really testing is the conjunction of the claim we started with, and all of these background assumptions
- As a result, when we perform tests we are testing whole theories, and it is the whole theory which gets confirmed or disconfirmed by the test

(Quine argues for his holism in 'Two Dogmas of Empiricism')

Putting the Argument All Together

(1) We should believe our best scientific theories

(2) Mathematics is indispensable to our best scientific theories

(3) Our theories get confirmed or disconfirmed **as whole theories**

∴ (4) We should believe that there are some mathematical objects

Putting the Argument All Together

> *if one is a realist about the physical world, then one wants to say that the Law of Universal Gravitation makes an objective statement about bodies — not just about sense data or meter readings. What is the statement? It is just that bodies behave in such a way that the quotient of two numbers associated with the bodies is equal to a third number associated with the bodies. But how can such a statement have any objective content at all if numbers and 'associations' (i.e. functions) are alike mere fictions?*

Putting the Argument All Together

> *It is like trying to maintain that God does not exist and angels do not exist while maintaining at the very same time that it is an objective fact that God has put an angel in charge of each star and the angels in charge of each of a pair of binary stars were always created at the same time! If talk of numbers and 'associations' between masses, etc. and numbers is 'theology' (in the pejorative sense), then the Law of Universal Gravitation is likewise theology.*
>
> *(Hilary Putnam, 'What is Mathematical Truth' in his Philosophical Papers vol. 1, pp. 74–5)*

Objecting to the Argument

- If you wanted to block this argument for platonism, you would have to reject one of the premises
  (1) We should believe our best scientific theories
  (2) Mathematics is indispensable to our best scientific theories
  (3) Our theories get confirmed or disconfirmed as whole theories
- As I said, (1) seems incontrovertible, but some philosophers have rejected (2) and (3)
  - Hartry Field rejected (2) in his *Science without Numbers*
  - Mary Leng rejected (3) in her *Mathematics and Reality*
- We won't have a chance to look at Leng's rejection of (3), but we will look at Field's rejection of (2) next week

An Answer to Benacerraf's Dilemma?

- But I want to end this week's lecture by pointing out that **if** we accept the Quine-Putnam Indispensability Argument, **then** we seem to have the resources to answer the Benacerraf Dilemma
- On the Quine-Putnam view, we know truths about mathematics **in exactly the same way** as we know any theoretical, scientific claims:
  - We come up with scientific theories which then get confirmed as a whole
  - This gives us reason to accept all the sentences in the scientific theories, **including ones about numbers**

An Answer to Benacerraf's Dilemma?

- However, we should acknowledge that this answer to Benacerraf's Dilemma comes with a cost
- We are used to thinking of mathematical truths as special: they can be known **a priori**
- But on the Quine-Putnam picture, they no longer look a priori
- Our mathematical beliefs are confirmed in exactly the same way as all of our theoretical beliefs

An Answer to Benacerraf's Dilemma?

- This was a result that Quine was more than happy with
  - According to Quine, the difference between mathematics and our empirical beliefs is just a matter of degree
  - We are reluctant to revise our mathematical beliefs because they are entrenched parts of so many theories
  - But we could revise our mathematical beliefs if we thought that was the best way of fixing a disconfirmed theory
- But although Quine was happy with this, we might not be!

For the Seminar

- In this week's seminar we are going to focus on Benacerraf's Problem, and Maddy's attempt to solve it by dragging mathematical objects into the causal world
- Required reading:
  - Benacerraf's 'Mathematical Truth', available through the VLE
  - Maddy's *Realism in Mathematics*, chs 1 & 2, available through the VLE

For Next Week

- We will be looking at Field's fictionalism
- Required reading:
  - *Shapiro* ch. 9