The Foundations of Mathematics
Lecture Five

Gödel's Incompleteness Theorems

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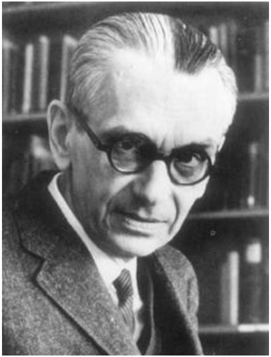
Kurt Gödel

- Gödel is one of the most important intellectual figures of the 20th Century
- His Incompleteness Theorems were monumental achievements in mathematical logic
- They had important technical consequences for subjects like computer theory
- But they also have huge consequences for the philosophy of mathematics



Kurt Gödel

Gödel's Incompleteness Theorems

Some Properties of Theories

The First Incompleteness Theorem

The Second Incompleteness Theorem

A Philosophical Consequence of Gödel's Theorems

Taking Care: Syntax versus Semantics

What is a Theory?

- In formal logic, a **theory** is any deductively closed set of sentences
- What does it mean to call a set of sentences "deductively closed"?
  - Imagine you have a set of sentences, Γ, and now consider all of the sentences you can logically deduce from Γ
  - Now imagine that you add all of those sentences into the set Γ
  - Γ would then be **deductively closed**
- Γ is deductively closed $=_{df}$ for every sentence ϕ, $\phi \in \Gamma \equiv \Gamma \vdash \phi$

An Example

- Imagine we start with the set of the following set of sentences

$$\left\{ \begin{array}{c} \text{Frege was a logicist} \\ \text{Russell was a logicist} \end{array} \right\}$$

- We can deduce lots of sentences from this set:
  - Frege was a logicist & Russell was a logicist
  - Frege was a logicist $\supset$ Russell was a logicist
  - $\exists x(x \text{ was a logicist})$
  - ...
- If we put all of these consequences in one big set with our two original sentences, then we would have a deductively closed set of sentences, i.e. a theory
- We call the sentences in a theory the **theorems** of that theory

Property One: Consistency

- One of the most important properties a theory can have is **consistency**
  - Γ is consistent $=_{df}$ there is no sentence ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$
- We are all used to the idea that a good theory should be consistent
- Most of us take it for granted that only consistent theories can be **true**

Property Two: Negation-Completeness

- Another good property for a theory is **negation-completeness**
  - Γ is negation-complete $=_{df}$ for every sentence ϕ, either $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$
- If a theory is negation complete, then it "decides" every sentence in the language of the theory
  - it either proves that sentence ($\Gamma \vdash \phi$)
  - or it refutes that sentence ($\Gamma \vdash \neg\phi$)
- A negation-complete theory leaves nothing out (hence "complete")

Property Three: Effectively Axiomatisable

- Γ **effectively axiomatisable** just in case there is a computer programme which could list all of the sentences in Γ without ever listing a sentence not in Γ
  - Since there are infinitely many sentences in a theory, the computer programme would never **finish** listing the sentences in Γ
  - Rather, the point is this: for any sentence in Γ, the computer programme will list this sentence after a finite period of time
- A decent theory should be effectively axiomatisable: otherwise, it is hard to imagine how simple, finite minds like ours could ever grasp it
- Even very powerful theories in mathematics (like modern set theory) are effectively axiomatisable

Property Four: Including Robinson Arithmetic

- **Robinson Arithmetic** is a very simple theory of arithmetic

  (1) $\forall x(0 \neq Sx)$
  (2) $\forall x \forall y(Sx = Sy \supset x = y)$
  (3) $\forall x(x \neq 0 \supset \exists y(x = Sy))$
  (4) $\forall x(x + 0 = x)$
  (5) $\forall x \forall y(x + Sy = S(x + y))$
  (6) $\forall x(x \times 0 = 0)$
  (7) $\forall x \forall y(x \times Sy = (x \times y) + x)$

- Γ includes Robinson Arithmetic just in case (1)–(7) are members of Γ

  (That's a bit of a simplification: all we really need is that Γ "interprets" (1)–(7); don't worry about that for now)

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  (3) $\forall x(x \neq 0 \supset \exists y(x = Sy))$
  (4) $\forall x(x + 0 = x)$
  (5) $\forall x \forall y(x + Sy = S(x + y))$
  (6) $\forall x(x \times 0 = 0)$
  (7) $\forall x \forall y(x \times Sy = (x \times y) + x)$

- Robinson Arithmetic is just a very small fragment of arithmetic, and so even fairly weak mathematical theories include Robinson Arithmetic

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All Four Properties Together?

- We've listed four good properties that a theory can have:
  (i) Consistent
  (ii) Effectively Axiomatisable
  (iii) Includes Robinson Arithmetic
  (iv) Negation-Complete

- It is natural to wonder whether a theory could have **all four** of these properties
- In particular, we would surely want our theory of arithmetic to have them all!

Gödel's First Incompleteness Theorem

- There is no theory Γ which has all four of the following properties:
  (i) Consistent
  (ii) Effectively Axiomatisable
  (iii) Includes Robinson Arithmetic
  (iv) Negation-Complete

How Did Gödel Do It!?

- Gödel managed to find a way of "coding up" claims about bits of language (like sentences and proofs, thought of as sequences of sentences) into claims about numbers
- This was a work of genius, but is not so shocking today: we are all familiar with the idea that computers "code" information into numbers
- Gödel then demonstrated that if Γ is effectively axiomatisable and includes Robinson Arithmetic, then we can construct a sentence, G, which is the code of the following claim:
  - Γ cannot prove G

How Did Gödel Do It!?

- Having constructed this sentence, Gödel proved that if Γ is consistent, then:
  - $\Gamma \nvdash G$
  - $\Gamma \nvdash \neg G$
- The actual proof of this result is difficult, but it is not too shocking that a sentence which said of itself that it is unprovable cannot be proven or refuted!

Gödel's First Incompleteness Theorem (Again)

- The upshot is that if Γ has these three properties:

  (i) Consistent
  (ii) Effectively Axiomatisable
  (iii) Includes Robinson Arithmetic

  then Γ cannot be

  (iv) Negation-Complete

- If Γ has (i)–(iii), then Gödel can construct his G, such that:
  - $\Gamma \nvdash G$
  - $\Gamma \nvdash \neg G$

What Happens if we Extend our Theory?

- You might wonder: what happens if we extend our theory Γ, either by adding G to it?
- Well, clearly, $\Gamma \cup \{G\} \vdash G$
- But now we'll be able to make a **new** sentence, G', which codes up the following claim:
  - $\Gamma \cup \{G\}$ cannot prove G'
- And then we'll prove:
  - $\Gamma \cup \{G\} \nvdash G'$
  - $\Gamma \cup \{G\} \nvdash \neg G'$
- So $\Gamma \cup \{G\}$ won't be negation-complete either!

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Gödel's Second Incompleteness Theorem

- If theory Γ has these three properties:
  (i) Consistent
  (ii) Effectively Axiomatisable
  (iii) Includes Robinson Arithmetic

  then Γ cannot prove that Γ is consistent:

  $$\Gamma \nvdash \mathsf{Con}_\Gamma$$

How Did Gödel Do It!?

- To say that Γ is consistent is just to say that there is no ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$
- So it is a claim about what sorts of proofs you can make from Γ
- This is exactly the sort of thing which Gödel has figured out how to code up in arithmetic!
- Lets use 'Con_Γ' as our shorthand for that code

How Did Gödel Do It!?

- Next we have to note that if Γ is consistent, then G is true
  - $\mathrm{Con}_\Gamma \supset G$
- This follows because if Γ is consistent, then it cannot prove G, and G says of itself that it cannot be proven by Γ!

How Did Gödel Do It!?

- And lastly, we need to note that (given some minimal background assumptions), Γ **itself** contains all the resources we need to prove Gödel's First Incompleteness Theorem
- Consequently, we have the following result:
  - $\Gamma \vdash \mathrm{Con}_\Gamma \supset G$
- And from here we can infer this:
  - $\Gamma \nvdash \mathrm{Con}_\Gamma$
- If Γ proved Con_Γ, then by modus ponens we would be able to show that $\Gamma \vdash G$, and we already know that that isn't true!

Gödel's Second Incompleteness Theorem (Again)

- Here is a concise statement on the Second Incompleteness Theorem:
  - If Γ is consistent, effectively axiomatisable and includes Robinson Arithmetic, then it cannot prove its own consistency
- Or to put it in a slightly more paradoxical way:
  - If an effectively axiomatisable theory which includes Robinson Arithmetic proves its own consistency, then that theory is not consistent!
- Compare that to this good advice for life:
  - If someone tells you that they are trustworthy, then they are not trustworthy!

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The Philosophical Significance of Gödel's Theorems

- It is hard to over-estimate the philosophical significance of Gödel's theorems
- They have huge consequences for pretty much **every** philosophy of maths
- We will quickly look at the absolutely devastating consequences that they had for Hilbert's Programme

Hilbert's Programme

- Hilbert divided mathematics into two parts: finitary mathematics (F) and ideal mathematics (I)
- Finitary mathematics was meaningful, and was all about the numerals that we use in maths, but ideal mathematics was just a game
- What makes ideal mathematics worthwhile is that it is a consistent extension of finitary mathematics ($I + F$ is consistent)
- Hilbert wanted to give a finitary proof that $I + F$ is consistent

David Hilbert

The Failure of Hilbert's Programme

- But Gödel's Second Incompleteness Theorem shows that this cannot be done
- F cannot prove that $I + F$ is consistent
  - If F could prove that $I + F$ is consistent, then it could prove that F is consistent
  - In symbols: $F \vdash \mathrm{Con}_F$
  - Hilbert's finitary mathematics was meant to be effectively axiomatisable and include Robinson Arithmetic
  - So the Second Incompleteness Theorem kicks in: if F really is consistent, then $F \nvdash \mathrm{Con}_F$!

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Syntax versus Semantics

- When we are thinking about a language **syntactically**, we are thinking of that language as a mere system of **signs**
- We do not care what the signs **mean**, or whether they mean anything at all

- When we are thinking about a language **semantically**, we are thinking about the language as a system of symbols with **meanings**
- For logical purposes, we are interested in things like: truth-values, references, satisfaction conditions

Two Notions of Entailment

- There are two notions of entailment:
  - Syntactic **deduction**, symbolised by $\vdash$
  - Semantic **consequence**, symbolised by $\vDash$
- $\Gamma \vdash \phi =_{df}$ we can write out a proof using sentences in Γ as premises, and ending with ϕ
- $\Gamma \vDash \phi =_{df}$ any interpretation which makes all the sentences in Γ true makes ϕ true too

Deductive Systems and Semantics

- Strictly speaking, it does not make sense just to say:
  - $\Gamma \vdash \phi$
- We have to also specify **which deductive system** we are using
  - Classical logic
  - Intuitionistic logic
  - ...

- Strictly speaking, it does not make sense just to say:
  - $\Gamma \vDash \phi$
- We have to say what **semantics** (i.e. theory of interpretations) we are using
  - Classical truth-theoretic semantics
  - Intuitionistic proof-theoretic semantics
  - ...

Soundness and Completeness

- A deductive system is **sound** relative to a semantics $=_{df}$
  for any sentence ϕ and set of sentences Γ: if $\Gamma \vdash \phi$ then $\Gamma \vDash \phi$
- A deductive system is **complete** relative to a semantics $=_{df}$
  for any sentence ϕ and set of sentences Γ: if $\Gamma \vDash \phi$ then $\Gamma \vdash \phi$
- In the dream scenario, our deductive system is sound **and** complete relative to our semantics
- Classical first-order logic is sound and complete relative to the standard semantics

Gödel's Theorems are Primarily Syntactic

- Gödel's Incompleteness Theorems are primarily **syntactic**
- They concern what can be **deduced** from certain theories

(1) If Γ is consistent, effectively axiomatisable and includes Robinson Arithmetic, then there is a sentence G such that: $\Gamma \nvdash G$ and $\Gamma \nvdash \neg G$

(2) If Γ is consistent, effectively axiomatisable and includes Robinson Arithmetic, then $\Gamma \nvdash \mathrm{Con}_\Gamma$

Gödel's Theorems can become Semantic

- However, if we assume that we are working with a complete deductive system, we can convert these syntactic results into semantic ones

(1) If Γ is consistent, effectively axiomatisable and includes Robinson Arithmetic, then there is a sentence G such that: $\Gamma \not\models G$ and $\Gamma \not\models \neg G$

(2) If Γ is consistent, effectively axiomatisable and includes Robinson Arithmetic, then $\Gamma \not\models \mathrm{Con}_\Gamma$

Second-Order Logic

IMPORTANT: NOT ALL DEDUCTIVE SYSTEMS ARE COMPLETE!

- A few times we have come up against the difference between **first-order logic** and **second-order logic**
- First-order logic lets us use quantifiers like this: $\exists x\ x$ is a philosopher
  - The variable x is in the position of a **singular term**, like 'Socrates' or 'Plato'
- Second-order logic lets us use quantifiers like this: $\exists F\ F(\text{Socrates})$
  - The variable F is in the position of a **predicate**, like 'is a philosopher' or 'is wise'

Second-Order Logic

- The standard deductive system for second-order logic is sound **but not complete** relative to the standard semantics
  - For all sentences ϕ and sets of sentences Γ: if $\Gamma \vdash \phi$ then $\Gamma \vDash \phi$
  - There is some sentence ϕ and some set of sentences Γ such that: $\Gamma \vDash \phi$ and $\Gamma \nvdash \phi$
- As a result, when we are dealing with a second-order system, we cannot convert Gödel's **syntactic** results into **semantic** ones

Second-Order Arithmetic

- In fact, we can formulate a theory of Arithmetic in second-order logic, known as **second-order Peano Arithmetic** (PA^2) with the following property:
  - For all sentences ϕ: either $PA^2 \vDash \phi$ or $PA^2 \vDash \neg\phi$
- In modern terminology, we say that PA^2 is **categorical**: categoricity is the semantic counterpart of negation-completeness
- And in fact, we even have this:
  - $PA^2 \vDash \mathrm{Con}_{PA^2}$
- So Con_{PA^2} is a **logical consequence** of PA^2, even though we cannot **syntactically deduce** Con_{PA^2} from PA^2

For the Seminar

- In the seminar we are going to look at Gödel's own platonist philosophy of mathematics
- Required reading:
  - Gödel, 'What is Cantor's Continuum Problem?', in *B&P*
- It may also be helpful to look at the following secondary material on Gödel's Theorems:
  - Giaquinto, *The Search for Certainty*, Part V

For Next Week

- We will start looking at some contemporary work in the philosophy of maths
- Required reading:
  - *Shapiro* ch. 8