The Foundations of Mathematics
Lecture Four

Hilbert's Programme

Rob Trueman
rob.trueman@york.ac.uk

University of York

Hilbert's Programme

A Reminder of why Mathematics is Special

- In Lecture 1, we pointed out that mathematical truths seems to have some special properties:
  - Mathematical truths are **necessarily** true
  - Mathematical truths can be known **a priori**
  - Mathematical truths can be known with **certainty**
- Add on top of all of this that mathematics seems to deal with **infinities**: there are infinitely many natural numbers, even more real numbers, **even more** sets of real numbers...
- Any good philosophy of maths needs to say something about these special features of mathematics

Introducing Formalism

- According to **formalism**, mathematics isn't really about "numbers" or "sets" (let alone **infinitely many** of them)
- Mathematics is just about manipulating **symbols**
- So if there is any such things as mathematical "truths" (a big **if**), then they are just truths about how we can manipulate symbols

Introducing Formalism

- Formalism is attractive for (at least) two reasons:
- **First:** it seems to fit very well with the actual practice of mathematics
  - Mathematicians seem to spend most of their time manipulating formulae and equations
- **Second:** it promises to deflate the big philosophical questions about mathematics
  - There is no mysterious necessary, a priori certain truth about numbers and the like; there are just simple truths about which symbols can be derived from which symbols

Two Types of Formalism

- **Term formalism:** mathematics is about **symbols**, like the **numerals** '2' and '3'
- **Game formalism:** mathematics is not **about** anything; mathematics is a meaningless **game** that we play with symbols
- These two versions of formalism are very different
  - According to term formalism, there **are** such things as mathematical truths; they are just truths about symbols
  - According to game formalism, there **are no** mathematical truths; mathematics isn't in the business of expressing propositions, true or false

A Problem for Term Formalism

- It is generally assumed that languages only contain a **denumerable infinity** of (types of) symbol
  - Remember: a set is **denumerable** iff it is no bigger than the set of natural numbers, $\{0, 1, 2, 3...\}$
- So if we think that mathematics is just about the symbols in a language, then mathematics can only deal with denumerable infinities
- But modern mathematics routinely deals with **non-denumerable** infinities, e.g. the real numbers

A Problem for Game Formalism

- According to game formalism, mathematics is just a game we play with symbols
- The big challenge for game formalism is to explain why the mathematical game that we play can be **applied** so usefully to the real world
- If mathematics is just a game, then it is in some non-compulsory: we **could** have played a different game
- For example, rather than playing our game of arithmetic, we could have played a variant in which 5+2=8
- Why is it that we can use our game of arithmetic to make bridges which can stand up, but we cannot use the above variant to make bridges?

A Problem for Game Formalism

> *an arithmetic without thought as its content will also be without possibility of application. Why can no application be made of a configuration of chess pieces? Obviously, because it expresses no thought. If it did so and every chess move conforming to the rules corresponded to a transition from one thought to another, applications of chess would also be conceivable.*

A Problem for Game Formalism

Why can arithmetical equations be applied? Only because they express thoughts. How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules? It is applicability alone which elevates arithmetic from a game to the rank of a science

(Frege 1903, Grundgesetze vol.2, §91)

Hilbert's Programme

David Hilbert

- Hilbert was one of the greatest mathematicians of the 19th and 20th Centuries
- He was also one of the most influential formalists
- He tried to combine term formalism with game formalism to make something stronger

David Hilbert

Hilbert's Suspicion of the Infinite

- Hilbert was deeply suspicious of the infinite, and that suspicion had two roots:
- **The Logical Paradoxes**
  - Early developments of infinitary set theory were beset by paradoxes (e.g. Russell's Paradox)
  - This raised the question: How can we be confident about this new branch of mathematics?
- **No Completed Infinities in Nature**
  - Hilbert was at bottom a good old Kantian: he thought that all meaningful mathematics was somehow derived from our intuitions of space and time
  - But nowhere in our intuitions do we ever come across completed, infinity totalities

Hilbert's Suspicion of the Infinite

> *the infinite is nowhere to be found in reality. It neither exists in nature nor provides a legitimate basis for rational thought.*
>
> *(Hilbert 1926, p.201)*

Hilbert's Love of Modern Mathematics

- Despite his suspicion of the infinite, Hilbert thought that modern, infinitary mathematics was one of humanity's greatest intellectual achievements

  "mathematical analysis is a symphony of the infinite" (Hilbert 1926, p.187)

  "No one shall drive us out of the paradise which Cantor has created for us." (ibid, p.191)

- How can Hilbert reconcile these two opposed attitudes?

Half-Term, Half-Game Formalism

- Hilbert divided mathematics into two broad categories:
- **Finitary mathematics**
  - The core of mathematics which is in no way committed to the existence of a completed infinity (i.e. infinitely big collections)
- **Ideal mathematics**
  - All the rest of mathematics
- Hilbert was a term formalist about finitary mathematics, and a game formalist about ideal mathematics

Half-Term, Half-Game Formalism

- The idea was that finitary mathematics was a body of **truths** about the symbols we use in mathematics, about their properties and the relations they bear to each other
- Ideal mathematics is then a **meaningless game** we play on top of the finitary mathematics
- But although ideal mathematics is meaningless, it is not **pointless**: ideal mathematics is useful because it helps us to derive results in **finitary mathematics**

Half-Term, Half-Game Formalism

- If Hilbert's Programme can be made to work, then it solves the two big problems we presented for term formalism and game formalism:

- Term formalists cannot deal with the branches of mathematics that concern non-denumerable infinities
- But Hilbert can: he simply says that those branches of mathematics are branches of **ideal** mathematics

- Game formalists cannot explain why mathematics is applicable in the real world
- But Hilbert can: ideal mathematics may only be a game, but the point of the game is to help us derive results in meaningful **finitary** mathematics

Hilbert's Programme

What are Hilbert's Terms?

- According to Hilbert, finitary mathematics is a body of truths about mathematical terms
- He suggested that instead of Arabic numerals like '2' and '3', we use finite strings of strokes as our numerals:
  - $1 =_{df} |$
  - $2 =_{df} ||$
  - $3 =_{df} |||$

  (Hilbert's numerals were a bit like Roman numerals, but he never used numerals like 'V' or 'C': it's just longer and longer strings of '|')
- To be clear, Hilbert's numerals were **types**, not **tokens**: Hilbert's numeral '|||' is the abstract type which all the token ink marks have in common

The Basic Finitary Truths

- The basic finitary truths are particular identities and non-identities, like the following:
  - $|| = ||$
  - $||| > ||$
  - $|| + | = |||$

- Here is how you are meant to understand them:
  - The symbol '$||$' is the same as the symbol '$||$'
  - The symbol '$||$' is a proper part of the symbol '$|||$'
  - If you write the symbol '$||$' followed by the symbol '$|$', then you get the symbol '$|||$'

The Basic Finitary Truths

- How do we know these basic finitary truths?
- They are meant to be something we can directly intuit
- We can just **see** that '||' is a proper part of '|||'
  - This gets a bit more complicated when we remember that Hilbert's numerals are types, not tokens
  - You cannot literally **see** types; you can only see **tokens**
  - Presumably, the idea is that our direct intuition of token numerals gives us an indirect access to the type numerals

Existential Generalisations

- In mathematics, we do not just make particular claims about particular numbers, we also make existentially generalised claims, like:
  (1) There is a prime number greater than 50
  (2) There is a prime number between 50 and 60
- (1) is known as an **unbounded** generalisation, because it simply tells us that there is a prime number somewhere in the **infinity** of numbers that comes after 50
- (2) is a **bounded** generalisation, because it sets an upper bound on where we need to look for a prime number greater than 50: it says that we will find one if we check the numbers up to 60

Existential Generalisations

- Unbounded existential generalisations **do not belong to finitary mathematics**
  - They are inherently infinitary
  - If you are looking for a prime number greater than 50 and you haven't found one yet, there is no guarantee that there won't be one further up in the series

- Bounded existential generalisations **do belong to finitary mathematics**
  - They are inherently finitary
  - You could re-write the claim that there is a prime number between 50 and 60 as a finite disjunction: either 50 is prime, or 51 is prime, or 52 is prime... or 59 is prime or 60 is prime

Universal Generalisations

- As well as existential generalisations, we also make universal generalisations in mathematics
  - $a + 1 = 1 + a$
- In ordinary, classical mathematics, we understand this as a universal generalisation over **infinitely many** numbers
- Obviously, there is no place for generalisations like that in **finitary** mathematics

Universal Generalisations

- But we can still use universal generalisations so long as we understand them in the right way
- On the finitary reading, '$a + 1 = 1 + a$' is a **schema**
- A schema like this isn't really the sort of thing you can **assert**
- Rather, to get something assertible, you must replace 'a' with a particular numeral
- So when you "assert" '$a + 1 = 1 + a$', you are really taking on the commitment to substitute any numeral you happen to come across into this schema
- The reason these sorts of schemas count as finitary is that you can describe a finite proof procedure for proving any given instance of the schema

Universal Generalisations

- One of the really important things to notice about finitary schema is that they cannot be **negated**
- If '$a + 1 = 1 + a$' were an ordinary universal claim, then its negation would be an **unbounded** existential generalisation:
  - $\exists n(n + 1 \neq 1 + n)$
- But unbounded existential generalisations are not finitary
- Remember, the **finitary** schema '$a + 1 = 1 + a$' just signals a willingness to substitute any numeral you come across into the schema!
- As a result, it is meaningless to negate a finitary universal generalisation

Finitary Mathematics and Intuitionism

- We can now see that classical logic **does not hold** within finitary mathematics
- If ϕ is a finitary universal generalisation, then $\phi \vee \neg\phi$ will not be true
  - $\neg\phi$ will be the negation of a finitary universal generalisation, and so will be meaingless
  - As a result, the whole disjunction $\phi \vee \neg\phi$ will be meaningless
- Similar rules will have to be restricted, like Double Negation Elimination and Reductio Ad Absurdum
- This is obviously very reminiscent of intuitionism, but we mustn't take the analogy too far

Finitary Mathematics and Intuitionism

- Brouwer's intuitionism was based on a particular metaphysical picture of numbers as **mental constructions**
- Hilbert's finitary mathematics treats numbers as types of symbol

- Although intuitionists refuse to assert some instances of $\phi \vee \neg\phi$, they accept that every instance of the schema is **meaningful**
- Hilbert says that some instances of $\phi \vee \neg\phi$ are **meaningless**

- The intuitionists think that we should give up on classical logic
- Hilbert thought that we only need to restrict classical logic within finitary mathematics; we can use classical logic in **ideal** mathematics

Hilbert's Programme

Introducing Formalism

Introducing Hilbert's Programme

Finitary Mathematics

Ideal Mathematics

The Limits of Finitary Mathematics

- Finitary mathematics makes up a **very small** fragment of mathematics as it is practised today
- It is also very difficult to use: you cannot use classical logic, and you cannot even negate universal generalisations
- This is where **ideal** mathematics comes in...

Ideal Elements

- The name 'ideal mathematics' is meant to draw an analogy with something quite familiar in mathematics
- Mathematicians have often introduced 'ideal' elements to make their theories easier to use
- These ideal elements are in some sense not what the mathematician is interested in, but adding them into the system makes it much easier for them to talk and think about the things they **are** interested in
- Two examples:
  - **Ideal points at infinity in geometry**: these points make it easier to talk about the relations between "real" lines
  - **Imaginary numbers (e.g.$\sqrt{-1}$):** these numbers make it easier to talk about the roots of equations

Ideal Mathematics

- Hilbert went one step further, and said that **all mathematics** which goes beyond the finitary is ideal
- For Hilbert, this means that non-finitary mathematics is just a meaningless game we play with symbols
- The point of playing this game is that it helps us to prove meaningful results in finitary mathematics

Conservativeness

- Of course, there are limits on what kinds of ideal games we are allowed to play
- In modern terminology, the ideal mathematics must be **conservative** over finitary mathematics

- Let I be our ideal mathematics, and let F be our finitary mathematics
- I is conservative over F iff:

  If $I + F \vdash \phi$, then $F \vdash \phi$,
  where ϕ is any claim of **finitary** mathematics

Explaining Conservativeness

- The idea is that we can use ideal mathematics to make it easier to prove claims in finitary mathematics
- But crucially, the ideal mathematics will not allow us to prove any finitary claim which we could not have proved **just by using the finitary mathematics**
- The ideal mathematics does let us prove any **new** finitary results; it just makes it **easier** to prove what we could already prove in finitary mathematics

From Conservativeness to Consistency

- We can actually simplify things a little bit
- Finitary mathematics is (we can assume) **negation complete**:
  - For any finitary claim ϕ, either $F \vdash \phi$ or $F \vdash \neg\phi$
- Given this assumption, the requirement that I be conservative over F just becomes the requirement that I be **consistent** with F
  - $I + F$ must be consistent

The Requirement of a Consistency Proof

> *There is just one condition, albeit an absolutely necessary one, connected with the method of ideal elements. That condition is a proof of consistency, for the extension of the domain by the addition of ideal elements is legitimate only if the extension does not cause contradictions to appear in the old, narrower domain, or, in other words, only if the relations that obtain among the old structures when the ideal structures are deleted are always valid in the old domain.*
>
> *(Hilbert 1926, p.199)*

Proving Consistency

- Now it turns out that we can explain what it means for $I + F$ to be consistent in **finitary** terms
- That's because formal proofs are just **finitary** objects, and the claim that $I + F$ is consistent is a finitary claim about those finitary proofs:
  - No valid proof which starts from $I + F$ ends with '0=1'

    (This is just a finitary universal generalisation over proofs, and can be understood in Hilbert's schematic way)
- **But very importantly:** the proof that $I + F$ is consistent **must** be given in **finitary terms**
  - It would be illegitimate to give an infinitary (i.e. ideal) proof, because the whole point of the proof is to establish that infinitary methods are legitimate!

Gödel's Incompleteness Results

- Hilbert was supremely confident that he could give a finitary proof of the consistency of ideal mathematics
- And in fact, Hilbert and his school did start to make some genuinely impressive progress on this front
- But in 1931, Kurt Gödel published his Incompleteness Theorems
- The Second Incompleteness Theorem stated that no consistent system powerful enough to express finitary arithmetic could prove its own consistency, let alone the consistency of $F + I$
- The consistency proof that Hilbert was looking for was impossible
- More on Gödel's Incompleteness Theorems next week!

For the Seminar

- Please read the following two items:
  - Hilbert 'On the infinite', in Benacerraf and Putnam *Philosophy of Mathematics: Selected Readings*
  - Giaquinto *The Search for Certainty*, Part IV Chapters 3 and 4
- You can find the Giaquinto chapters on the VLE, as well as information about where to find the Hilbert article

For the Next Lecture

- We are going to be discussing the fundamentals of Gödel's Incompleteness Theorems
- Please read the following in advance:
  - Nagel and Newman (2001) *Gödel's Proof*, ch. 7