The Foundations of Mathematics

Lecture Three

Intuitionism

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Intuitionism

The Last Two Lectures

- Over the last two lectures, we have been focussing on **logicism**, the idea that mathematics is logic in disguise
- Although logicism is **undeniably** attractive, it runs into serious trouble when it comes to the infinite ontology of mathematics
- In this lecture, we are going to look at a very different way of thinking about mathematics, called **intuitionism**

Kant on Mathematics

- Intuitionism has its roots in Kant's philosophy of mathematics
- So we will start by very quickly outlining Kant's view

Immanuel Kant

Two Distinctions

- Kant drew two important distinctions
- **The Analytic/Synthetic Distinction**
  - A truth is **analytic** iff it is true purely by virtue of its meaning

    (In Kant's words: the subject is contained in the predicate)
  - A truth is **synthetic** iff it is true partly by virtue of its meaning, and partly by virtue of how the world is
- **The A Priori/A Posteriori Distinction**
  - A truth is **a priori** iff we can know that it is true without relying on experience
  - A truth is **a posteriori** iff we the only way to know that it is true is via experience

Two Distinctions

	Analytic	**Synthetic**
A Priori	All bachelors are male	2+5=7
A Posteriori	X	All whales are mammals

Mathematics as A Priori Synthetic

- According to Kant, mathematics is **a priori synthetic**
  - By contrast, Frege thought that it was **a priori analytic**, and in fact Frege was explicit that his logicism was a reaction against Kantianism
- So we can know mathematics **without** experience, but the truths of mathematics are not made true just by their meaning, but also by how the world is
- This can be very hard to understand: how can we know a synthetic claim about how the **world** is without relying on experience?

A Super Quick Sketch of Kant's Picture

- According to Kant, the world we experience is not the world as it is in itself
  - Kant called the world as we experience it the **phenomenal** world
  - And he called the world as it is in itself the **noumenal** world
- The phenomenal world is what you get when our minds impose their concepts on the noumenal world
- There are certain basic concepts which we **have to** apply to the noumenal world, called **forms**
- So we can discover facts about the phenomenal world by looking at these forms
- This is where **a priori synthetic** knowledge comes from

A Helpful Analogy

- Here is a helpful analogy from Moore (2012: p. 119–20)
- For Kant, it is like we always look at the world through special conceptual spectacles
- One way to find out how the world looks through these spectacles is by **looking through them**
- Stuff that we can only find out by looking through the conceptual spectacles is **a posteriori synthetic**

A Helpful Analogy

- But another thing we can do is **look at how the spectacles work**: by looking at how they work, we might be able to figure out that the world is guaranteed to look a certain way when we look through them
  - If we look at some spectacles and see that they have rose-tinted lenses, then we will know that the world will look rose-tinted through those spectacles
- Stuff that we can find out by looking at the conceptual spectacles is **a priori synthetic**

Mathematics as A Priori Synthetic

- So for Kant, mathematics was the sort of thing you could learn about by looking at our conceptual spectacles
- In particular, Kant thought that we had to apply the forms of space and time on the world, and that is where mathematics comes from
- Roughly, our grasp of the sequence of numbers comes from our grasp on sequences in space and time

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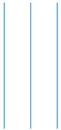

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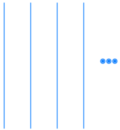


Some Problems for the Kantian View

- There are lots of objections to raise against this Kantian view
- There are general philosophical worries:
  - Kant's view is a grand metaphysic, according to which we (the "transcendental subjects") are separate from the world as it really is, and somehow make the world as we experience it out of how the world is
  - It is not hard to imagine lots of philosophers worrying about this kind of picture!

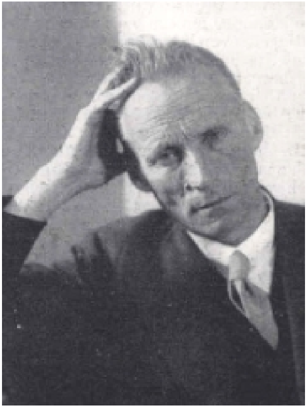
Some Problems for the Kantian View

- But there are more particular worries too:
  - Kant's conviction that space and time were forms was in part built on the assumption that Euclidean geometry was the only possible geometry, but we now know that to be false
  - Many branches of mathematics seem to have moved far away from intuition (e.g. transfinite arithmetic) — what can Kant say about them?

Intuitionism

L.E.J. Brouwer

- The first great intuitionist was Brouwer
- Intuitionism developed out of the Kantian story we just told
- It accepted some elements of Kantianism, but rejected others



L.E.J. Brouwer

Numbers as Mind Dependent

- According to Brouwer, numbers are **mind dependent** entities
- They are things that minds **construct**
- Numbers are in no sense part of the external world

Numbers as Mind Dependent

In the words of Brouwer's student, Arend Heyting:

> *"mathematics is a production of the human mind [...]*
>
> *we do not attribute an existence independent of our thought, i.e., a transcendent existence, to the integers or to any other mathematical objects [...]*
>
> *mathematical objects are by their very nature dependent on human thought."*

Heyting, 'The intuitionist foundations of mathematics', in B&P

How do we Construct Numbers?

- The way that Brouwer thought that we constructed numbers was very Kantian
- He thought that we started with the basic form of time
  - Brouwer didn't want to work with the form of space, because he thought that non-Euclidean geometry had made that a less firm foundation
- We grasp the **natural** numbers by grasping the step from one unit of time to two units of time, then on to three units of time, and so on
- We then grasp the continuum of **real** numbers by grasping the notion that between any two moments of time, there is an inexhaustible continuum of moments

Mathematics as Inherently Mental

- It is important to emphasise that for Brouwer, mathematics is an inherently mental, **private** practice
- Mathematics is primarily something you do in your head: you construct various numbers with various properties
- When you write out a mathematical claim, or a mathematical proof, you are just trying to communicate the construction you did in your head
- What is more, this linguistic communication is imperfect, it looses something; all you can hope is that what you write leads your reader to perform the same construction as you in their head

Mathematics as Inherently Mental

As the meaning of a word can never be fixed precisely enough to exclude every possibility of misunderstanding, we can never be mathematically sure that the formal system expresses correctly our mathematical thoughts.

(Heyting, 'Disputation' in B&P p.69)

Mathematics as an Activity

- In a way, it is not even quite right for Brouwer to talk about **mathematical** truth
- It is representations (e.g. sentences, propositions etc) that are true or false
- But for Brouwer, mathematics isn't about **representing** anything
- It's about **constructing** things, and so it isn't clear that truth really comes in
  - Compare: when a carpenter builds a table, we don't say that the construction of that table is **true**
- Again, what you are doing when you write down a mathematical argument is giving your reader a prompt for performing a particular construction

Mathematics as an Activity

Strictly speaking the construction of intuitive mathematics in itself is an action and not a science

(Brouwer Collected Works, p.61n)

Objections...

- This is just a **sketch** of intuitionism, but it is already enough to raise lots of objections
  - How **exactly** do we construct numbers from our experience of time?
  - Who exactly is included in the **we** here? Just ordinary humans, like you and me, or idealised humans?
  - If mathematics is really a process of construction, why does it come so naturally to present mathematical **arguments**, rather than **instructions** for mathematical constructions?
- But we're going to set these sorts of worries largely to one side, and look more closely at what happens to logic and mathematics if we accept intuitionism

Intuitionism

The Law of Excluded Middle

- **The Law of Excluded Middle (LEM):** $\phi \vee \neg\phi$
  - Either Kant was human or Kant was not human
  - Either grass is green or grass is not green
  - Either the Sun orbits the Earth or the Sun does not orbit the Earth
  - ...

- **Bivalence:** Every (meaningful) sentence is true or false
- Bivalence is different from LEM, but they are obviously related
- In fact, they become equivalent if we accept the following two principles

  (T) 'p' is true iff p
  (F) 'p' is false iff $\neg p$

Rejecting LEM

- LEM is a classical law of logic, but intuitionists reject it
- Consider Goldbach's Conjecture: every even number greater than 2 is the sum of two primes
- No counterexample to Goldbach's Conjecture have ever been found, but we have no proof of it
- Now abbreviate Goldbach's Conjecture as G, and consider the following instance of LEM:

$$G \vee \neg G$$

- If mathematical objects exist independently of us, then surely this disjunction is true: either all the even numbers conforms to Goldbach's Conjecture, or there is some even number out there breaking the conjecture, waiting to be found

Rejecting LEM

- But things are different if we think about numbers in Brouwer's intuitionist way
- Roughly, this is how Brouwer understands Goldbach's Conjecture:
  - $G =_{df}$ we have have a method of showing that every even number is the sum of two primes
- Roughly, this is how Brouwer understands the **negation** of Goldbach's Conjecture:
  - $\neg G =_{df}$ we have a method of constructing a counterexample to G, i.e. an even number which is not the sum of two primes
- When they are understood in this way, there is no guarantee that either G or $\neg G$: it might be that we do not have a method of proving or of refuting G

Rejecting LEM

- This led Brouwer to reject LEM as being based on a mistaken conception of what numbers are
- **A VERY IMPORTANT POINT:**
  - To say that intuitionist reject LEM is just to say that there are some instances of $(\phi \vee \neg\phi)$ which they **do not assert**
  - It is **not** to say that they assert some instances of $\neg(\phi \vee \neg\phi)$
  - **No intuitionist ever asserts any instance of this!!!**
  - In intuitionistic logic, $\neg(\phi \vee \neg\phi)$ is a **contradiction** (just like in classical logic)

Intuitionistic Logic

- The details of **intuitionistic logic** were worked out by Brouwer's student Heyting
- Normally, we base our semantics on the idea of **truth**
  - $\phi \vee \psi$ is true iff ϕ is true or ψ is true (or both)
- Heyting realised that this only works if we think that we are dealing with a domain of things which exist independently of us
- If we are dealing with things which only exist if we construct them, like numbers according to the intuitionists, then we need to base our semantics on the idea of **proof**
- The idea is that we only **assert** a (mathematical) sentence if we have a **proof** for it

Intuitionistic Logic

- A proof $\phi \,\&\, \psi$ consists of a proof of ϕ and a proof of ψ
- A proof of $\phi \vee \psi$ consists either of a proof of ϕ, or of a proof of ψ (or of both)
- A proof of $\phi \supset \psi$ consists of a method for converting a proof of ϕ into a proof of ψ
- A proof of $\neg\phi$ consists of a method for converting a proof of ϕ into a proof of an absurdity (e.g. 0=1)
- A proof of $\forall n \phi(n)$ consists of a method that, given any number n, will produce a proof of $\phi(n)$
- A proof of $\exists n \phi(n)$ consists of the construction of a number n and a proof that $\phi(n)$

Rejecting LEM (again)

- Given this semantics, it is easy to see why an intuitionist would not assert $G \vee \neg G$
- We can only assert this disjunction if we have a proof of it
- Such a proof would consist either of a proof of G or a proof of $\neg G$
- We do not have a proof of either G or $\neg G$
  - A proof of G would consist of a method which, given any n, would show that if n is even then n is the sum of two primes
  - A proof of $\neg G$ would consist of a method of converting any proof of G into a proof of an absurdity
- So we cannot assert $G \vee \neg G$
  - **But remember:** we also cannot assert $\neg(G \vee \neg G)$!!!

Rejecting DNE

- LEM is not the only rule which goes, a few others do too
- **Double Negation Elimination (DNE):** $\neg\neg\phi \vdash \phi$
- This is a classical law, but the intuitionists reject it
- All we need to assert $\neg\neg\phi$ is a method of turning any proof of $\neg\phi$ into a proof of an absurdity
- Clearly, we can have that **without** having any proof of ϕ
- So we can be in a position to assert $\neg\neg\phi$ without being in a position to assert ϕ

Rejecting RAA

- Intuitionists accept one version of **reductio ad absurdum** (RAA):
  - If you can derive a contradiction from the supposition ϕ, then infer $\neg\phi$
- In fact, intuitionists do more than merely accept this version of (RAA), it gives us the fundamental way of proving that $\neg\phi$!
- But intuitionists **reject** this (classical) version of (RAA):
  - If you can derive a contradiction from the supposition $\neg\phi$, then infer ϕ
- Clearly, we can have a way of converting any proof of $\neg\phi$ into a proof of absurdity **without** having a way of proving ϕ
- Observation: if intuitionists had accepted (DNE), then the intuitionistic version of (RAA) would have led to the classical version

Quantifiers and Negations

- In classical logic, $\neg\forall n\phi(n)$ entails $\exists n\neg\phi(n)$
- So in classical logic, you can prove that a certain number without a certain property **exists** just by proving that a universal generalisation is false
- But no intuitionistic logic: we can have a method of converting any proof of $\forall n\phi(n)$ into a proof of absurdity **without** having a way of constructing a number n which can then be proved to be $\neg\phi$
- This fits exactly with the intuitionist conception of numbers as dependent objects:
  - Numbers only exist by construction!

Brouwer on Intuitionistic Logic

- Heyting's intuitionistic logic had a mixed reception
- Brouwer was unimpressed, because it focussed on logic and language, rather than what maths is really about: **constructing** numbers **in your mind**
- Heyting had great sympathy with Brouwer's worry, and thought that intuitionistic logic was little more than a linguistic aide to mental construction

Heyting's Lasting Legacy

- However, in the long run, Heyting's intuitionistic logic has been the real lasting legacy of intuitionism
- Very few philosophers subscribe to Brouwer's metaphysical conception of numbers as mental constructs, but lots of logicians care about intuitionistic logic
  - In fact nowadays, when people use the word 'intuitionism', they just mean the intuitionistic logic
- That's because there are lots of arguments that we should reject classical logic and favour intuitionistic logic which have nothing to do with Brouwer's metaphysic
- The great modern proponent of intuitionism in this sense was Dummett, and I heartily recommend that anyone who likes really good philosophy read Dummett
  - You can find Dummett's 'The philosophical basis of intuitionistic logic' in *B&P*

Intuitionism

What is Intuitionist Mathematics?

- In this final part of the lecture, we will look at Brouwer's **intuitionistic mathematics**
- By 'intuitionistic mathematics', I mean the mathematics which Brouwer thought made sense according to his metaphysical conception of numbers as mental objects
- As we will see, this intuitionistic mathematics is very different from **classical mathematics**
- **Warning:** by 'intuitionistic mathematics', some modern day writers may just mean mathematics done with a background intuitionistic logic

The Classical Hierarchy of Infinities

- According to classical mathematics, there is a hierarchy of different sized infinities
- The smallest infinity is the (cardinal) number of natural numbers: 0, 1, 2, 3...
- But there are bigger infinities too: the (cardinal) number of **real** numbers is larger than the (cardinal) number of natural numbers
- And there are bigger infinities than that, and bigger than that, and bigger than that...
- Terminology: a set is **denumerable** iff it is no bigger than the set of natural numbers

The Intuitionist on Infinity

- Intuitionists believe in infinity: the numbers are somehow constructed out of our intuition of the temporal succession of moments, and we seem to be able to intuit that there need be no limit to those moments
- But it is very hard to see how an intuitionist could ever get to any infinity **bigger** than the smallest (i.e. to non-denumerable sets)

"the intuitionist recognises only the existence of denumerable sets"

(Brouwer, 'Intuitionism and formalism', in B&P p. 81)

Potential versus Actual Infinity

- We can get clearer on the intuitionist conception of infinity by distinguishing **potential** infinity from **actual** (or "completed") infinity
- To say that the numbers are **potentially infinite** is to say:

  if you have a finite collection of numbers, then no matter how big that finite collection is, there are more numbers out there to be collected
- To say that the numbers are **actually infinite** is to say:

  you can actually form one infinitely big collection of all the numbers

Potential versus Actual Infinity

- Classical logic is happy with actual infinities, and it is only by dealing with actual infinities that they get to bigger and bigger infinities
- Intuitionistic logic is only happy with potential infinity: we have to construct the numbers, and we could never construct a completed, infinitely big collection
- So the intuitionists must reject all the classical mathematics dealing with non-denumerable infinities as meaningless

What about the Real Numbers?

- According to classical logic, there are more real numbers than there are natural numbers
- This must be a mistake for the intuitionist. But what should an intuitionist say about real numbers?

- Classical mathematics builds real numbers out of rational numbers with Dedekind-cuts
  - Roughly: take a series of rational numbers which includes no greatest number; the least upper bound of that series is a real number
  - We can then use such a series of rational numbers to represent the real which is its least upper bound

What about the Real Numbers?

- Intuitionists can use a similar method to construct their real numbers
- **BUT:** because intuitionists do not believe in actual infinities, they can never fully specify a sequence of rational numbers which includes no greatest number
- All they can do is specify some initial, finite segment of the sequence
- They can do this in one of two ways:
  - By following a rule which tells us what the next number in the sequence will be
  - By making a free choice at each moment about what the next number in the sequence will be

What about the Real Numbers?

- This leads to **substantial** differences between classical analysis and intuitionistic analysis
- And this is not just a matter of the classical mathematician proving **more** results than the intuitionistic mathematician
- Intuitionistic analysis **contradicts** classical analysis
- For example, a result in intuitionistic analysis is that every function from the reals to the reals is continuous
  - See *Shapiro* pp. 181–4 for details

Reception by Mathematicians

- Intuitionism is philosophically fascinating — there are some features which are attractive, and some consequences which are not
- However, few mathematicians were won over by intuitionism
- Most mathematicians still happily use classical logic, infinitary mathematics, and use the standard real numbers

Reception by Mathematicians

- For many mathematicians (and philosophers), this is a **pragmatic** decision
  - Classical mathematics is incredibly fruitful, and proven itself useful for science, and it strikes us as a somehow **natural** way of doing maths
  - Intuitionistic mathematics is not so useful, and it takes a lot of work to think like an intuitionist (although the work may be worth it!)
- But I will leave it to you to decide how much these pragmatic considerations should count for!

For the Seminar

- Required reading:
  - Brouwer, ‘Consciousness, philosophy of mathematics’
  - Heyting ‘Disputation’
  - Both are available in *B&P*
- Optional further reading:
  - Brown *Philosophy of Mathematics*, chapter 8