The Foundations of Mathematics
Lecture Three

Intuitionism

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Intuitionism

Kant's Philosophy of Mathematics

The Fundamentals of Intuitionism

Intuitionistic Logic

Cutting-Edge 19th Century Mathematics

Intuitionistic Mathematics

The Last Two Lectures

- Over the last two lectures, we have been focussing on **logicism**, the idea that mathematics is reducible to logic
- Although logicism is *undeniably* attractive, it runs into serious trouble when it comes to the infinite ontology of mathematics
- In this lecture, we are going to look at a very different way of thinking about mathematics and infinity called **intuitionism**

Kant on Mathematics

- Intuitionism has its roots in Kant's philosophy of mathematics
- So we will start by very quickly outlining Kant's view

Immanuel Kant

Two Kantian Distinctions

- **The Analytic/Synthetic Distinction**
  - A truth is **analytic** iff it is true purely by virtue of its meaning
    (In Kant's words: the subject is contained in the predicate)
  - A truth is **synthetic** iff it is true partly by virtue of its meaning, and partly by virtue of how the world is
- **The A Priori/A Posteriori Distinction**
  - A truth is **a priori** iff we can know that it is true without relying on experience
  - A truth is **a posteriori** iff we the only way to know that it is true is via experience

Orthogonal Kantian Distinctions

	Analytic	**Synthetic**
A Priori	All bachelors are male	2+5=7
A Posteriori	**X**	All whales are mammals

Mathematics as A Priori Synthetic

- According to Kant, mathematics is **a priori synthetic**
  - By contrast, Frege thought that it was **a priori analytic**; Frege explicitly presented his logicism as a reaction against Kantianism
- So we can know mathematics **without experience**, but the truths of mathematics are not made true just by their meaning, but also by how **the world** is
- This can be very hard to understand: how can we know a synthetic claim about how **the world** is without relying on **experience**?

A Super Quick Sketch of Kant's Picture

- The **phenomenal** world (i.e. the world as we experience it) is distinct from the **noumenal** world (i.e. the world as it is in itself)
- The phenomenal world is a joint production: mind meets (noumenal) world
- We impose certain **forms** of thought on the phenomenal world
- So we can discover facts about the phenomenal world by looking at these forms
- This is where **a priori** **synthetic** knowledge comes from

A Helpful Analogy

- Here is a helpful analogy from Moore (2012: p. 119–20)
- For Kant, it is like we always look at the world through special intellectual spectacles
- One way to find out how the world looks through these spectacles is by **looking through them**
- Stuff that we can only find out by looking through the intellectual spectacles is **a posteriori synthetic**

A Helpful Analogy

- But another thing we can do is **look at how the spectacles work**
- By looking at how they work, we might be able to figure out that the world is guaranteed to look a certain way when we look through them
  - If we look at some spectacles and see that they have rose-tinted lenses, then we will know that the world will look rose-tinted through those spectacles
- Stuff that we can find out by looking at the intellectual spectacles is **a priori** **synthetic**

Mathematics as A Priori Synthetic

- So for Kant, mathematics was the sort of thing you could learn about by looking at our intellectual spectacles
- In particular, Kant thought that we had to apply the forms of space and time on the world, and that is where mathematics comes from
- Roughly, our grasp of the sequence of numbers comes from our grasp on sequences in space and time

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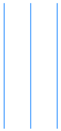

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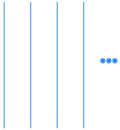

Some Problems for the Kantian View

- **General worry about Kantianism**
  - Kant's view is a grand metaphysic, according to which we (the "transcendental subjects") are separate from the world as it really is, and somehow make the phenomenal world out of the noumenal world
- **Specific worries about Kant's philosophy of maths**
  - Kant thought that the forms of space and time were Euclidean, but modern physics uses non-Euclidean spacetimes
  - Many branches of mathematics seem to have moved far away from intuition (e.g. transfinite arithmetic) — what can Kant say about them?

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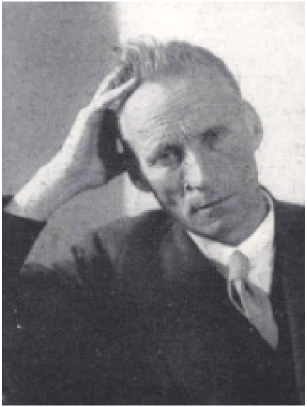
L.E.J. Brouwer

- The first great intuitionist was Brouwer
- Intuitionism developed out of the Kantian story we just told
- It accepted some elements of Kantianism, but rejected others



L.E.J. Brouwer

Numbers as Mind-Dependent

- **Platonists** think of numbers as **mind-independent** abstract objects
- According to Brouwer, numbers are **mind-dependent** objects
- They are things that minds **construct**
- Numbers have no properties except by our construction

Numbers as Mind-Dependent

In the words of Brouwer's student, Arend Heyting:

> *"mathematics is a production of the human mind [...] we do not attribute an existence independent of our thought, i.e., a transcendent existence, to the integers or to any other mathematical objects [...] mathematical objects are by their very nature dependent on human thought."*
>
> *Heyting, 'The intuitionist foundations of mathematics', in B&P*

How do we Construct Numbers?

- Brouwer thought that we constructed numbers from the basic form of time
  - Brouwer didn't want to work with the form of space, because he thought that non-Euclidean geometry had made that a less firm foundation

- We grasp the **natural** numbers by grasping the step from one unit of time to two units of time, then on to three units of time, and so on

- We then grasp the continuum of **real** numbers by grasping the notion that between any two moments of time, there is an inexhaustible continuum of moments

Mathematics as Inherently Mental

- It is important to emphasise that for Brouwer, mathematics is an inherently mental, **private** practice
- Mathematics is primarily something you do in your head: you construct various numbers with various properties
- When you write out a mathematical claim, or a mathematical proof, you are just trying to communicate the construction you did in your head
- What is more, this linguistic communication is imperfect, it loses something; all you can hope is that what you write leads your reader to perform the same construction as you in their head

Mathematics as Inherently Mental

As the meaning of a word can never be fixed precisely enough to exclude every possibility of misunderstanding, we can never be mathematically sure that the formal system expresses correctly our mathematical thoughts.

(Heyting, 'Disputation' in B&P p.69)

Mathematics as an Activity

- In a way, it is not even quite right for Brouwer to talk about **mathematical truth**
- It is representations (e.g. sentences) that are true or false
- But for Brouwer, mathematics isn't about **representing** anything
- It's about **constructing** things, and so it isn't clear that truth really comes in
- A mathematical proof is an instruction manual for a particular construction
  - Compare: when a carpenter builds a table, we don't say that the construction of that table, or the instruction manual that the carpenter followed, is *true*

Mathematics as an Activity

Strictly speaking the construction of intuitive mathematics in itself is an action and not a science

(Brouwer Collected Works, p.61n)

Objections...

- This is just a *sketch* of intuitionism, but it is already enough to raise lots of objections
  - How **exactly** do we construct numbers from our experience of time?
  - Who exactly is included in the **we** here? Just ordinary humans, like you and me, or idealised humans?
  - If mathematics is really a process of construction, why does it come so naturally to present mathematical **arguments**, rather than explicit **instructions** for mathematical constructions?

- But, in this lecture, we're going to set these sorts of worries largely to one side, and look more closely at what happens to logic and mathematics if we accept intuitionism

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The Law of Excluded Middle

- **The Law of Excluded Middle (LEM):** $\mathcal{A} \vee \neg\mathcal{A}$
  - Either Kant was human or Kant was not human
  - Either grass is green or grass is not green
  - Either the Sun orbits the Earth or the Sun does not orbit the Earth
- LEM is a law of Classical Logic
- **Bivalence:** Every (meaningful) sentence is true or false
- Bivalence is equivalent to LEM, if we accept these principles:

(T) '$\mathcal{A}$' is true iff $\mathcal{A}$

(F) '$\mathcal{A}$' is false iff $\neg\mathcal{A}$

LEM and Platonism

- **Goldbach's Conjecture**
  - Every even number greater than 2 is the sum of two primes
- No counterexample to Goldbach's Conjecture have ever been found, but we have no proof of it
- Nonetheless, platonists still insist that Goldbach's Conjecture is true or false, and assert the corresponding instance of LEM: $G \vee \neg G$
- For a platonist, Goldbach's Conjecture might just be an **infinite coincidence**

Intuitionists Against LEM

- Roughly, this is how an intuitionist understands Goldbach's Conjecture:
  - $G =_{df}$ there is a finite method which, when applied to any natural number n, shows that, if n is an even number greater than 2, then n is the sum of two primes

- Roughly, this is how an intuitionist understands the **negation** of Goldbach's Conjecture:
  - $\neg G =_{df}$ there is a finite method of constructing a counterexample to G, i.e. an even number greater than 2 which is not the sum of two primes

- Understood in this way, there is **no guarantee** that G or $\neg G$: there might be no finite method of proving or refuting G

Rejecting LEM

- Brouwer rejected LEM as based on a mistaken conception of what numbers are
- To say that intuitionist reject LEM is just to say that there are some instances of $\mathcal{A} \vee \neg\mathcal{A}$ which they **do not assert**
- It is not to say that they **assert** some instances of $\neg(\mathcal{A} \vee \neg\mathcal{A})$
- In Intuitionistic Logic, $\neg(\mathcal{A} \vee \neg\mathcal{A})$ is a **contradiction** (just like in Classical Logic)

From Truth to Proof

- The details of **Intuitionistic Logic** were worked out by Brouwer's student Heyting
- Normally, we base our semantics on the idea of **truth**
  - $\mathcal{A} \vee \mathcal{B}$ is true iff $\mathcal{A}$ is true or $\mathcal{B}$ is true (or both)
- Heyting claimed that this only works for platonists
- Intuitionists, who think that numbers only have the properties they are constructed as having, should base their semantics on **proof**
- The idea is that we only **assert** a (mathematical) sentence if we have a **proof** for it

Intuitionistic Logic

- A proof $\mathcal{A} \wedge \mathcal{B}$ consists of a proof of $\mathcal{A}$ and a proof of $\mathcal{B}$
- A proof of $\mathcal{A} \vee \mathcal{B}$ consists either of a proof of $\mathcal{A}$, or of a proof of $\mathcal{B}$ (or of both)
- A proof of $\mathcal{A} \rightarrow \mathcal{B}$ consists of a finite method for converting a proof of $\mathcal{A}$ into a proof of $\mathcal{B}$
- A proof of $\neg\mathcal{A}$ consists of a finite method for converting a proof of $\mathcal{A}$ into a proof of an absurdity (e.g. 0=1)
- A proof of $\forall n \mathcal{A}(n)$ consists of a finite method that, given any number n, will produce a proof of $\mathcal{A}(n)$
- A proof of $\exists n \mathcal{A}(n)$ consists of the construction of a number n and a proof that $\mathcal{A}(n)$

Rejecting LEM (again)

- According to an intuitionist, we can only assert $G \vee \neg G$ if we have a proof of it
- This proof would consist either of a proof of G or a proof of $\neg G$
- We do not have a proof of either G or $\neg G$
  - A proof of G would consist of a finite method which, given any n, would show that, if n is even and greater than 2, then n is the sum of two primes
  - A proof of $\neg G$ would consist of a method of converting any proof of G into a proof of an absurdity
- So we cannot assert $G \vee \neg G$
  - **But remember:** we also cannot assert $\neg(G \vee \neg G)$!!!

Rejecting DNE

- LEM is not the only rule which goes, a few others do too
- **Double Negation Elimination (DNE):** $\neg\neg\mathcal{A} \vdash \mathcal{A}$
- This is a Classical Law, but the intuitionists reject it
- All we need to assert $\neg\neg\mathcal{A}$ is a method of turning any proof of $\neg\mathcal{A}$ into a proof of an absurdity
- Clearly, we can have that without having any proof of $\mathcal{A}$

Intuitionistic versus Classical RAA

- Intuitionists accept one version of **reductio ad absurdum (RAA)**:
  - If you can derive a contradiction from the supposition $\mathcal{A}$, then infer $\neg\mathcal{A}$

- In fact, intuitionists do more than merely accept this version of (RAA), it gives us the fundamental way of proving that $\neg\mathcal{A}$!

- But intuitionists *reject* this Classical version of (RAA):
  - If you can derive a contradiction from the supposition $\neg\mathcal{A}$, then infer $\mathcal{A}$

- Clearly, we can have a way of converting any proof of $\neg\mathcal{A}$ into a proof of absurdity without having a way of proving $\mathcal{A}$

Quantifier Conversion

- In Classical Logic, $\neg\forall n\mathcal{A}(n)$ entails $\exists n\neg\mathcal{A}(n)$
- So in classical logic, you can prove that a certain number without a certain property **exists** just by proving that a universal generalisation is false
- But no intuitionistic logic: we can have a method of converting any proof of $\forall n\mathcal{A}(n)$ into a proof of absurdity without having a way of constructing a number n which can then be proved to be $\neg\mathcal{A}$
- This fits exactly with the intuitionist conception of numbers as dependent objects
  - Numbers only exist by construction!

Brouwer on Intuitionistic Logic

- Heyting's Intuitionistic Logic had a mixed reception
- Brouwer was unimpressed, because it focussed on logic and language, rather than what maths is really about: **constructing** numbers **in your mind**
- Heyting had great sympathy with Brouwer's worry, and thought that intuitionistic logic was little more than a linguistic aide to mental construction

Heyting's Lasting Legacy

- However, in the long run, Heyting's Intuitionistic Logic has been the real lasting legacy of intuitionism
- Very few philosophers subscribe to Brouwer's metaphysical conception of numbers as mental constructs, but lots of logicians care about Intuitionistic Logic
- That's because there are arguments that we should reject classical logic and favour Intuitionistic Logic which have nothing to do with Brouwer's metaphysic
- For a classic discussion of Intuitionist Logic, see Dummett's 'The philosophical basis of intuitionistic logic', in *B&P*

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Number Systems

- **Natural Numbers ($\mathbb{N}$):** $0, 1, 2, 3, 4 \ldots$
- **Integers ($\mathbb{Z}$):** $\ldots - 4, -1, -2, -1, 0, 1, 2, 3, 4 \ldots$
  - There seem to be twice as many integers as there are naturals
  - But it turns out you can put the integers and the naturals in 1-1 correspondence!
- **Rational Numbers ($\mathbb{Q}$):** $\frac{m}{n}$
  - There seems to be *lots* more rationals than naturals: between any two rationals, there are infinitely many rationals!!!
  - But it turns out you can put the rationals and the naturals in 1-1 correspondence!
- **Real Numbers ($\mathbb{R}$):** $3.14159265 \ldots$
  - Georg Cantor proved that it was *impossible* to put the reals and the naturals into 1-1 correspondence!

The Diagonalization Argument

- Imagine an attempt to pair off each real number between 0 and 1 with a natural number

$$\begin{array}{ccc} 1 & \rightarrow & 0.1508\ldots \\ 2 & \rightarrow & 0.2983\ldots \\ 3 & \rightarrow & 0.2497\ldots \\ 4 & \rightarrow & 0.0026\ldots \\ \ldots & \rightarrow & \ldots \end{array}$$

- Here is a number not on this list: $0.9119\ldots$

- **The General Rule:** the *nth* decimal in our new number is 1 if the *nth* decimal on the *nth* row of the table is 9; otherwise, it is 9

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Cantor's Theorem

- The cardinality of natural numbers is written as $\aleph_0$ (aka the only *denumerable* infinity)
- The cardinality of the real numbers can be proven to be $2^{\aleph_0}$
- So Cantor's Diagonalization Argument proves that $2^{\aleph_0} > \aleph_0$
- **Cantor's Theorem:** $2^{\kappa} > \kappa$, for any cardinality κ
- So there are *infinitely many* infinite cardinalities
- In fact, there are so many infinite cardinalities that it does not even make sense to talk about the cardinality of infinite cardinalities!!!

Discontinuity

- **Rough:** Function f is discontinuous at point x iff there is some value ϵ s.t. no matter how close y is to x, the gap between $f(x)$ and $f(y)$ is no smaller than ϵ
- **Precise:** Function f is discontinuous at point x iff $\exists\epsilon{>}0\,\forall\delta{>}0\,\exists y\big((0 < |x - y| < \delta) \wedge (|f(x) - f(y)| \geq \epsilon)\big)$

Everywhere Discontinuous

- There are functions from reals to reals which are discontinuous everywhere
- **The Dirichlet Function:** $D(x) = 1$ if x is rational, and $D(x) = 0$ if x is irrational
- For each x and $\delta > 0$, we can find some y s.t. $0 < |x - y| < \delta$ and $|f(x) - f(y)| \geq 1$
- That's because between any two real numbers, there is a rational number and an irrational number

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What is Intuitionist Mathematics?

- In this final part of the lecture, we will look at Brouwer's **Intuitionistic Mathematics**
- By 'Intuitionistic Mathematics', I mean the mathematics that Brouwer thought made sense according to his metaphysical conception of numbers as mental objects
- As we will see, Intuitionistic Mathematics rejects the two classical results we just reviewed
- **Warning:** by 'Intuitionistic Mathematics', some modern day writers may just mean mathematics done with a background Intuitionistic Logic

The Intuitionist on Infinity

- Intuitionists believe in infinity: the numbers are somehow constructed out of our intuition of the temporal succession of moments, and we seem to be able to intuit that there need be no limit to those moments

- But it is very hard to see how an intuitionist could ever get to any infinite cardinal larger than $\aleph_0$

"the intuitionist recognises only the existence of denumerable sets"

(Brouwer, 'Intuitionism and formalism', in B&P p. 81)

Potential versus Actual Infinity

- We can get clearer on the intuitionist conception of infinity by distinguishing **potential** infinity from **actual** (or "completed") infinity
- To say that the numbers are **potentially infinite** is to say:
  - If you have a finite collection of numbers, then no matter how big that finite collection is, there are more numbers out there to be collected
- To say that the numbers are **actually infinite** is to say:
  - You can actually form one infinitely big collection of all the numbers

All Potential, No Non-Denumerable!

- **Classical Mathematics** is happy with actual infinities, and it is only by dealing with actual infinities that they get to bigger and bigger infinities
- **Intuitionistic Mathematics** is only happy with potential infinity: we have to construct the numbers, and we could never construct a completed, infinitely big collection
- So the intuitionists must reject all the Classical Mathematics dealing with non-denumerable infinities as meaningless

Real Numbers in Classical Mathematics

- According to Classical Mathematics, there are more real numbers than there are natural numbers
- This must be a mistake for the intuitionist. But what should an intuitionist say about real numbers?
- Classical Mathematics builds real numbers out of rational numbers with **Dedekind-Cuts**
  - **Roughly:** Take a series of rational numbers which includes no greatest number; the least upper bound of that series is a real number
  - We can then use such a series of rational numbers to represent the real which is its least upper bound

Real Numbers in Intuitionistic Mathematics

- Intuitionists can use a similar method to construct their real numbers
- **BUT:** because intuitionists do not believe in actual infinities, they can never fully specify a sequence of rational numbers which includes no greatest number
- All they can do is specify some initial, finite segment of the Dedekind-Cut
- They can do this in one of two ways:
  - By following a rule which tells us what the next number in the sequence will be
  - By making a free choice at each moment about what the next number in the sequence will be

Everywhere Continuous

- **Classical:** Functions are mappings from *arguments* to *values*
- **Intuitionist:** A function is a procedure which yields more precise approximations of its value for more precise approximations of its argument
- This conception led Brouwer to his **Continuity Theorem**
  - All functions from reals to reals are everywhere continuous
- What about the **Dirichlet Function**?
  - $D(x) = 1$ if x is rational, and $D(x) = 0$ if x is irrational
- According to intuitionists, that function does not exist!
  - It takes LEM to guarantee that every real is either rational or irrational!

Reception by Mathematicians

- Intuitionism is philosophically fascinating — there are some features which are attractive, and some consequences which are not
- However, few mathematicians were won over by intuitionism
- Most mathematicians still happily use Classical Logic, infinitary mathematics, and use the standard real numbers

Pragmatism?

- For many mathematicians (and philosophers), this is a **pragmatic** decision
  - Classical Mathematics is incredibly fruitful, and proven itself useful for science, and it strikes us as a somehow **natural** way of doing maths
  - Intuitionistic Mathematics is not so useful, and it takes a lot of work to think like an intuitionist (although the work may be worth it!)
- But I will leave it to you to decide how much these pragmatic considerations should count for!

For the Seminar

- Required reading:
  - Brouwer, ‘Consciousness, philosophy of mathematics’
  - Heyting ‘Disputation’
  - Both are available in *B&P*
- Optional further reading:
  - Brown *Philosophy of Mathematics*, chapter 8