# The Foundations of Mathematics

Lecture One

# Frege's Logicism

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# Frege's Logicism

# Course Structure

- Contact Hours
  - 9 × 1 hour lectures (Monday 16:00–17:00)
  - 9 × 1.5 hour seminars (Friday)
  - Weekly Office Hour (Tuesday 14:30–15:30)
- Procedural Requirements
  - Attend lectures and seminars
  - Complete all required reading
  - Participate in seminar discussions

    (Please bring copies of seminar readings with you, as well as answers to at least one set question *in writing*)
- Assessment
  - 1 × 1,200 word essay, due 12 noon Monday Week 7, Spring Term (Formative)
  - 1 × 4,000 word essay, due Monday Week 1, Summer Term (Summative)

# Course Structure

- In Weeks 2–6 we will look at some of the most historically important projects in the foundations of mathematics:
  - Week 2: Frege's logicsm
  - Week 3: Russell's paradox and his own logicism
  - Week 4: Intuitionism
  - Week 5: Hilbert's programme
  - Week 6: Gödel's incompleteness theorems
- In Weeks 7–10 we will look at some more recent projects:
  - Week 7: The Quine-Putnam Indispensability Argument
  - Week 8: Fictionalism
  - Week 9: Structuralism
  - Week 10: Neo-Fregean logicism

# Readings

- Main textbook:
  - Shapiro, S (2000) *Thinking about Mathematics*
- Also recommended:
  - Benacerraf, P & Putnam, H eds (1983) *Philosophy of Mathematics: Selected Readins*, 2nd edition
  - Giaquinto, M (2002) *The Search for Certainty*
  - Shapiro, S (2005) *The Oxford Handbook of Philosophy of Mathematics and Logic*

# Readings

- Required weekly readings can be found in the Module Information Document, which is available via the VLE page for this module
- Where possible, required Seminar Readings will be made available on the VLE
- A general reading list can be found in the EARL list on the VLE, and suggestions for further readings will be made in the lectures

# Frege's Logicism

# What Makes Mathematics So Special?

- Mathematical truths, like 2+5=7, seem to have some special properties
- They are **necessarily** true
  - 2+5 is 7 in every possible world
  - 2+5 couldn't have been anything other than 7
- They can be known **a priori**
  - You do not need to do any experiments to check whether 2+5=7
  - You can **prove** that 2+5=7 with pen and paper
- They can be known with **certainty**
  - There is no doubt about whether 2+5=7
  - Once you prove something in mathematics, you can rely on it without worry

# Introducing Logicism

- Logical truths, like $\forall x(Fx \supset Fx)$, seem to be special in exactly the same ways
  - They are **necessarily** true
  - They can be known **a priori**
  - They can be known with **certainty**
- A natural thought: mathematical truths are just complicated logical truths
  - mathematical concepts can be defined in logical terms
  - we can derive all of mathematics from pure logic
- The idea that mathematics (or some suitably large portion of mathematics) can be derived from logic is called **logicism**

# Logic As Insubstantial

- Logicism is a deeply attractive and natural position, but things are not as simple as they may seem
- It is commonly thought that logic is in some sense 'insubstantial' or even 'contentless'
  - Logic places no demands on the world
  - Logical truths do not tell us anything about how the world is
  - If you tell me that it is either raining or it isn't, then you haven't told me how the weather is

# The Empty Ontology of Logic

- It is very hard to say what the insubstantiality of logic really amounts to, but it is tempting to say that this is part of it:
  - Logic is ontologically innocent
  - The truths of logic do not require that any particular objects exist
  - You can never use logic to prove that a particular object exists

# The Infinite Ontology of Arithmetic

- In contrast with logic, mathematics seems to bring with it an **infinitely big** ontology
- Throughout these lectures, we will focus for the most part on **arithmetic**, which studies the **natural numbers** $(0, 1, 2, ...)$ and the operations that can be performed on them (addition, multiplication...)
  - The reason we will focus on arithmetic is that it is a comparably simple branch of mathematics, but it has pretty much all of the philosophically interesting features of mathematics
- At least on the face of it, arithmetic is ontologically committed to the existence of infinitely many numbers

## The Infinite Ontology of Arithmetic

- In arithmetic, we appear to refer to numbers with singular terms, like 'the number 2' and 'the prime number between 6 and 8'
- But perhaps even more importantly, we appear to **quantify** over numbers too
- Not all the truths of arithmetic are as simple as 2+5=7
- We also have truths like this:
  - There is a prime number between 6 and 8
  - There are infinitely many prime numbers
- These look like existential claims, the first telling us that a certain prime number exists, and the second telling us that *infinitely* many prime numbers exist

# Logicism and the Ontology of Arithmetic

- Again and again in this course, we will see the infinite ontology of arithmetic causing trouble for various philosophies of mathematics
- In the case of logicism, a logicist can only do one of two things:
  (1) Deny that arithmetic is really committed to the existence of infinitely many numbers (or any numbers at all, for that matter)
  (2) Accept that arithmetic is committed to the existence of numbers, but then insist that pure logic can prove the existence of numbers after all
- In this lecture we are going to look at a logicism of type 2
- This was the logicism of Frege, one of the greatest philosophers ever to have lived

# Frege's Logicism

# Frege

- Frege invented modern quantificational logic
- He was also a brilliant philosopher of language (his distinction between sense and reference continues to drive philosophical thought today)
- He was **also** the first great logicist

# Grundlagen

- Frege published two treatises on logicism
- In the first, called *Die Grundlagen der Arithmetik*, Frege laid out his philosophical arguments for logicism
- He intended to work out all of the technical details of his logicism in his second treatise, *Die Grundgesetze der Arithmetik*
- The Grundgesetze was meant to come in three volumes, but in the end Frege only published two
  - More on why he gave up early next week!
- We're going to focus on Grundlagen, and in fact we're going to jump in half way through

# Introducing Hume's Principle

- In §63 of Grundlagen, Frege introduced the following principle, which has since become known as **Hume's Principle**:

(HP) The number of $F$s = the number of $G$s iff $F$ and $G$ are equinumerous

$$NxFx = NxGx \equiv F \sim G$$

- What do we mean when we say that $F$ and $G$ are 'equinumerous'?
- As a first approximation: each $F$ can be paired off with a $G$, and *vice versa*

# Introducing Hume's Principle

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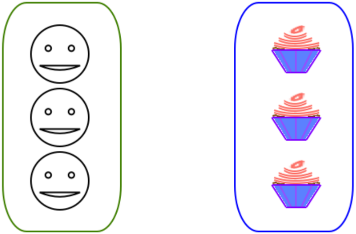

- The number of people = the number of cakes iff each person can be paired off with a cake, and *vice versa*

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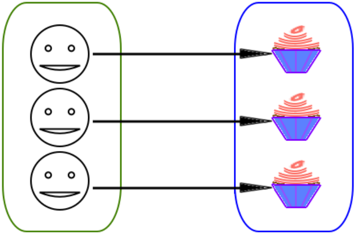

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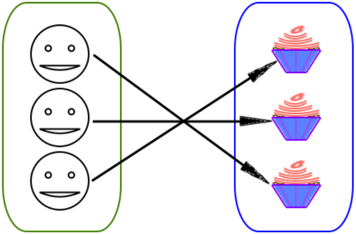

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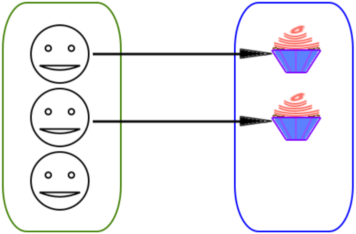

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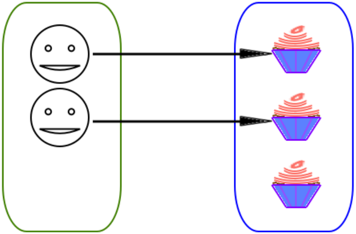

- The number of people ≠ the number of cakes, since one cake cannot be paired off with a person

# Hume's Principle as a Definition of Number Talk

(HP) $NxFx = NxGx \equiv F \sim G$

- This principle connects a claim about numbers ($NxFx = NxGx$) with a claim which doesn't mention any numbers at all ($F \sim G$)
- It is immediately tempting to try thinking of (HP) as a kind of **definition** of our number talk
- (HP) certainly isn't an ordinary, explicit, definition (e.g. vixen $=_{df}$ female fox)
- But it is still tempting to say that (HP) is another kind of definition, a **contextual** or **implicit** definition

# Re-carving Content

(HP) $NxFx = NxGx \equiv F \sim G$

- In Frege's words (Grundlagen §64), the idea is that in (HP), we take the content of an equinumerosity-claim, $F \sim G$, and **re-carve** it as the content of an identity claim, $NxFx = NxGx$
- So the left and right hand sides of (HP) have the same content, they just break it up in different ways
- If this idea could be made to work, then (HP) would surely count as a kind of definition

# Frege's Logicism

# Introducing Frege's Theorem

- It turns out that if we can count (HP) as a definition, then we can vindicate logicism (about arithmetic, at least)
- That's because logic+(HP) entails all of arithmetic, so if (HP) is a definition, then logic+definitions entails all of arithmetic
- The result that logic+(HP) entails all of arithmetic is now known as **Frege's Theorem**
- In what follows I will sketch how Frege used logic+(HP) to prove that there are infinitely many numbers
- For a full proof of Frege's Theorem, see: Wright 1983 chapter 4

# Defining Equinumerosity

- The first thing we need to do is define equinumerosity
- $F \sim G =_{df} \exists R$ :
  (i) $\forall x(Fx \supset \exists! y(Gy \,\&\, Rxy))$
  (ii) $\forall x \forall y((Fx \,\&\, Fy \,\&\, x \neq y) \supset \neg \exists z(Gz \,\&\, Rxz \,\&\, Ryz))$
  (iii) $\forall x(Gx \supset \exists! y(Fy \,\&\, Rxy))$
  (iv) $\forall x \forall y((Gx \,\&\, Gy \,\&\, x \neq y) \supset \neg \exists z(Fz \,\&\, Rxz \,\&\, Ryz))$
- Do not let this very complex definition of equinumerosity confuse you!
- The important thing to note is that in this definition, we used only **logical** vocabulary: quantifiers, variables, connectives and $=$

# Proving that Numbers Exists

- You can prove, using pure logic, that $F \sim F$, no matter what property $F$ is
- That means that logic+(HP) entails that $NxFx$ exists, no matter which property $F$ is
- Start by letting the $F$ and $G$ in (HP) be the same property:
  $$NxFx = NxFx \equiv F \sim F$$
- Since $F \sim F$ is a logical truth, logic+(HP) allows us to infer that $NxFx = NxFx$
- And from $NxFx = NxFx$, we can infer: $\exists y(y = NxFx)$
  - After all, only things which exist can be identical to anything!

# Proving that Infinitely Many Numbers Exist

- Frege now uses a clever trick to show that each of the **infinitely many** numbers exist
- We start by defining 0 as $Nx(x \neq x)$, i.e. the number of things which are not identical to themselves
  - Remember, we just saw that $Nx(x \neq x)$ is guaranteed to exist by logic+(HP)!
- This seems like a good definition for 0, because it is a logical truth that **nothing** is not identical to itself

# Proving that Infinitely Many Numbers Exist

- We then define 1 as $Nx(x = 0)$, i.e. the number of things which are identical to 0
  - This seems like a good definition for 1, because we have just used logic+(HP) to prove that 0 exists, and clearly, 0 is the **one and only** thing which is identical to 0
- We then define 2 as $Nx(x = 0 \vee x = 1)$, i.e. the number of things which are identical to 0 or 1
  - This seems like a good definition for 2, because we have just used logic+(HP) to prove that 0 and 1 exist, and clearly, 0 and 1 are the **only two** things which are identical to 0 or 1

# Proving that Infinitely Many Numbers Exist

- In general, we define $n+1$ as $Nx(x = 0 \vee x = 1 \vee \ldots \vee x = n)$
- Moreover, we can always use (HP) to prove that since $n$ exists, then so does $n+1$
- With a little bit of extra work, this can be converted into a proof that there are infinitely many numbers
  - We need to add a general definition of Number, and a general definition of Successor (the successor of $n$ is $n+1$)
  - Frege provided both of these, using purely logical terms, but we won't go into that here

# Frege's Logicism

# Logicism Vindicated?

- We have just seen that if (HP) counts as some sort of definition, then logicism is vindicated
  - Logic+(HP) entails all of arithmetic
- Unfortunately, after suggesting that we might think of (HP) as a kind of definition, Frege himself raised a serious objection to the idea
- This objection appears in §66 of Grundlagen, and is known as the **Julius Caesar Objection**

# The Julius Caesar Objection

- (HP) tells us how to figure out whether a sentence of the form '$NxFx = NxGx$' is true: its true just when the corresponding sentence of the form '$F \sim G$' is true
- But what about identity claims which are not of that form?
  (J) Julius Caesar $= Nx(x \neq x)$
- (HP) has now way to tell us whether or not (J) is true, i.e. whether or not Julius Caesar is the number 0
- And according to Frege, this is a devastating objection to the idea that (HP) is a definition

# Why does this Objection Matter?

- It's important not to misunderstand Frege's objection here
- Frege is not disappointed because he was really wondering whether Julius Caesar is the number 0, and was hoping that (HP) would tell him!
- Frege takes it that we all **know** that Julius Caesar is not a number
- The problem is that although we all know it, we don't know it from (HP)
- So, it seems, (HP) is not an adequate definition of our number talk: there is more to our concept than is contained in (HP)

# How Exactly should we Understand the Objection?

- That is the Julius Caesar Objection in outline, but the details are trickier
- Philosophers have offered lots of different interpretations of the objection, and you could fill a whole lecture talking about them
- We will look at this objection in more detail in Week 10, when we discuss neo-Fregean logicism, but for now I will just quickly sketch how I understand it

# My Interpretation of the Objection

- When we lay down (HP) as a definition, all we **really** define is the whole sentence '$NxFx = NxGx$'
- It is tempting to think that we somehow define the terms '$NxFx$' and '$NxGx$' as well, but we don't: the whole sentence gets a meaning all at once, but the component parts don't
- As a result, it doesn't make sense to try to substitute a term, like 'Julius Caesar', for '$NxFx$'

# Frege's Logicism

# What is a Logicist to do?

- Frege's Theorem tells us that logic+(HP) entails all of arithmetic
- If we could think of (HP) as a definition, then that would be enough to give us logicism (about arithmetic)
- But Frege thought that the Julius Caesar Objection demonstrated that (HP) couldn't act as a definition of our number terms
- Still, that does not mean we have to turn our back on Frege's Theorem
- If we could somehow show that (HP) **itself** could be derived from logic+definitions, then we could still use Frege's Theorem to vindicate logicism
- That is just what Frege tried to do

# Introducing Classes

- In §68 of Grundlagen, Frege introduced **extensions**, which nowadays are normally called **sets** or **classes**
- A class is a collection of objects:
  - The class of dogs is a collection containing all the dogs, and nothing else
  - The class of cats is a collection containing all the cats, and nothing else
  - The class of fat cats is a collection containing all the cats which are fat, and nothing else
- Some notation — we use curly brackets to refer to classes: $\{x : Fx\}$ is the class of $F$s

# What are Classes?

- In Grundlagen, Frege is frustratingly quiet about what classes are, or why we are allowed to appeal to them in a logicist account of arithmetic

  > "I assume that it is known what the extension of a concept is" (Grundlagen §68 n. 1)

- It is clear that Frege thinks that classes are a special kind of logical object, but he doesn't explain what that amounts to in Grundlagen
- Frege does say a bit more in Grundgesetze, and we will look at that next week
- For now, we will follow Frege's lead, and just take the notion of a class for granted

# Frege's Explicit Definition of Numbers

- By helping himself to his classes, Frege was finally able to offer an explicit definition of the numbers which satisfied him
- For Frege, numbers were classes, but not classes of ordinary things: they were classes **of properties**
- In particular, $NxFx$ is the class of all properties which are equinumerous with $F$
  - In symbols: $NxFx =_{df} \{G : G \sim F\}$

# Frege's Explicit Definition of Numbers

- Take for example $Nx$($x$ is a surviving member of the Beatles)
- There are exactly two surviving members of the Beatles, so this number is the class of all properties which exactly two things have
- Here are some properties we would find in that class:
  - *$x$ is a surviving member of the Beatles*
  - *$x$ is a planet closer to the Sun than the Earth*
  - *$x$ is a prime number between 4 and 8*

# The End...?

- From this definition of numbers in terms of classes, and some sensible assumptions about how classes behave, you can derive (HP):

(HP) $NxFx = NxGx \equiv F \sim G$

- So if classes do count as 'logical objects', as Frege thought, then at last, Frege's logicism is vindicated!
- Unfortunately, things didn't end so happily
- In the Grundgesetze, Frege states his key assumption about classes (which he labelled 'Basic Law V'):

(V) $\{x : Fx\} = \{x : Gx\} \equiv \forall x(Fx \equiv Gx)$

- And as we will see next week, (V) turned out to be inconsistent!

# For the Seminar

- Required Reading:
  - Gottlob Frege, 'The Concept of Number' (Grundlagen §§55–109)
- You can find these sections of Grundlagen under Readings in the Course Materials section of the VLE page for this module
- You can also find them in Benacerraf and Putnam's *Philosophy of Mathematics: Selected Readings*

# References


- Frege, G (1884) *Die Grundlagen der Arithmetik* (Breslau: Koebner)
- Wright, C (1983) *Frege's Conception of Numbers as Objects* (Aberdeen University Press)