Paradoxes

Bonus Slides

Vagueness: Fuzzy Logic

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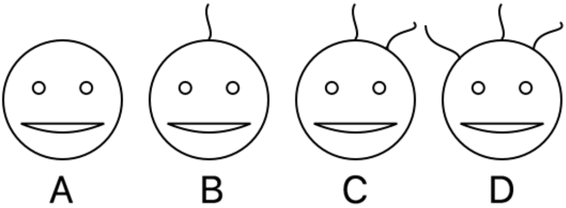
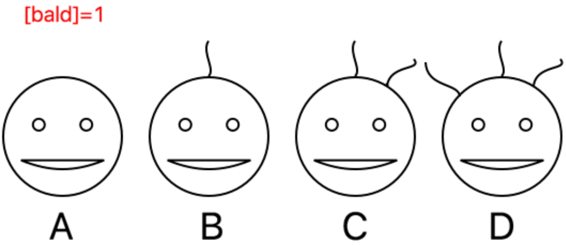
Vagueness: Fuzzy Logic

More or Less True



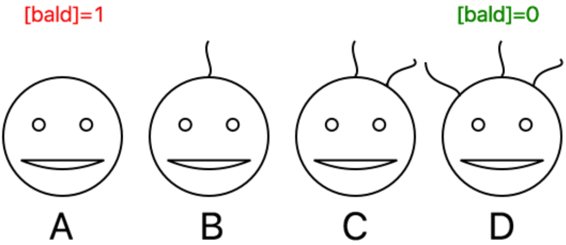
- When dealing with vagueness, it is natural to talk about "degrees of truth"

More or Less True



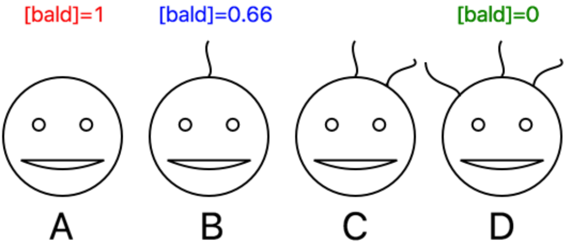
- It is completely true to say that A is bald

More or Less True



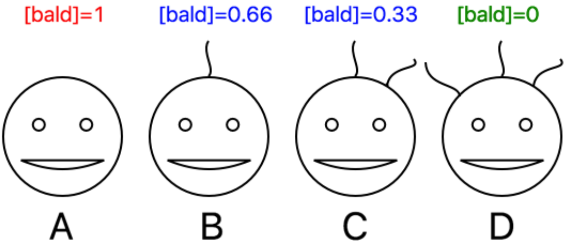
- It is completely false to say that D is bald

More or Less True



- It is not quite completely true to say that B is bald

More or Less True



- It is even less true to say that C is bald

Degrees of Truth

- Some logicians take this way of speaking, "more or less true" very seriously
- They say that there are many truth values other than **true** (1) and **false** (0)
- There are lots of different ways of trying to fill this idea out
- I am going to focus on one standard way, sometimes known as **fuzzy logic**
  - However, for a potentially more interesting alternative, see (Edgington 1996)

Continuum-Many Truth Values

- There are **continuum**-many truth values, which are represented with real numbers between 0 and 1
- 0 represents complete falsehood, 1 represents complete truth, and the closer we get to 1, the closer we get to truth
- A bit of notation: $[A]$ is the truth-value of the sentence A
  - If A is completely true, then $[A] = 1$
  - If A is completely false, then $[A] = 0$
  - If A is somewhere in between, then $[A]$ is some number between 1 and 0, e.g. 0.75

A Fuzzy Semantics

$$
\begin{array}{lcl}
[\sim A] & = & 1 - [A] \\
[A \,\&\, B] & = & \min\{[A], [B]\} \\
[A \vee B] & = & \max\{[A], [B]\} \\
[A \supset B] & = & 1 \text{ if } [B] \geq [A] \\
 & & 1 - ([A] - [B]) \text{ otherwise} \\
[\forall x A x] & = & \text{glb}\{[A^{x/a}]\text{: for all } a\} \\
[\exists x A x] & = & \text{lub}\{[A^{x/a}]\text{: for all } a\}
\end{array}
$$

(See Sainsbury and Williamson 1997, p. 476)

Validity in this Fuzzy Logic

- This semantics tells us how to decide which truth-values complex sentences get, in terms of the truth-values that the simple sentences get
- But how do we decide whether an argument is **valid** in this logic?
- In classical logic, we say this:
  - An argument is valid iff there is no interpretation which makes the premises true and the conclusion false
- But what should we say about interpretations that make the premises and/or conclusion somewhere between completely true and completely false?

Validity in this Many-Valued Logic

- On one standard account, we say this:
  - An argument is valid iff there is no interpretation which assigns the conclusion a lower truth value than the argument's least true premise
- In classical logic, validity preserves truth
- In this fuzzy logic, validity preserves degree of truth
  - But see (Edgington 1996) for an interesting alternative!

Vagueness: Fuzzy Logic

Form B

- Again, let's start with Form B of the Sorites paradoxes

  A man with 0 hairs is bald

  If a man with 0 hairs is bald, then a man with 1 hair is bald

  If a man with 1 hair is bald, then a man with 2 hairs is bald

  ...

  If a man with 9,999 hairs is bald, then a man with 10,000 hairs is bald

  Therefore a man with 10,000 hairs is bald

- What truth-value do these premises get?

The Truth-Values of the Premises

- Surely [A man with 0 hairs is bald] $= 1$
- And let's say [A man with 1 hair is bald] $= 0.95$
- In that case: [If a man with 0 hairs is bald then a man with 1 hair is bald] $= 1 - (1 - 0.95) = 0.95$
- So at least one of the conditional premises is not completely true

The Truth-Value of the Conclusion

- Does this mean that if we buy fuzzy logic, we can dismiss this paradox as premise-flawed?
- No: in fuzzy logic, it is still paradoxical if premises with very high truth-values lead to a conclusion with a very low truth-value
- But that is just what happens in Sorites Paradoxes
  - We can suppose that [A man with 10,000 hairs is bald] $= 0$

Modus Ponens is Invalid

- The real solution to the Sorites paradoxes in this fuzzy logic is to say that they are **fallacious**
- This might come as a surprise, because the only rule of inference used in this paradox is **modus ponens**
  - $A \supset B; A; \therefore B$
- But in the fuzzy logic we have described, modus ponens really is invalid

Modus Ponens is Invalid!

- We said [A man with 1 hair is bald] $= 0.95$
- Let's also suppose that [A man with 2 hairs is bald] $= 0.9$
- In that case: [If a man with 0 hairs is bald then a man with 2 hairs is bald] $= 1 - (0.95 - 0.9) = 0.95$
- So applying modus ponens here would take us to a conclusion with a lower truth value than any of the premises, which means the rule is not valid
  - [If a man with 1 hair is bald then a man with 2 hairs is bald] $= 0.95$
  - [A man with 1 hair is bald] $= 0.95$
  - [A man with 2 hairs is bald] $= 0.9$

Form A

- Form A of the paradox is also invalid:

  A man with 0 hairs is bald

  (QP) $\forall n$ [a man with n hairs is bald $\supset$ a man with $n+1$ hairs is bald]

  Therefore $\forall n$[a man with n hairs is bald]

- But I will leave it as an exercise for you to figure out how!

Vagueness: Fuzzy Logic

Modus Ponens is Invalid!

- We just saw that the many-valued logic above solves the Sorites paradoxes by rejecting modus ponens as invalid
- But this is a pretty radical thing to do
  - Modus ponens is a **very** basic rule of inference!
- Admittedly, we don't have to reject all uses of modus ponens
- Modus ponens still works fine when applied to sentences that are completely true or completely false
- But vagueness is everywhere, and if we represent vagueness with truth values between 1 and 0, then we will almost never be able to use modus ponens

Modus Ponens is Invalid!

- This is not the only objection to fuzzy logic
- Here are two more:
  - Fuzzy logic seems to get some truth-values wrong
  - Fuzzy logic has no obvious way of dealing with higher-order vagueness

Odd Truth Values

- Suppose Paul is an exact borderline case of being bald, so that [Paul is bald] $= 0.5$
- In that case: [Paul is not bald] $= 1 -$ [Paul is bald] $= 1 - 0.5 = 0.5$
- So [Paul is bald and Paul is not bald] $= 0.5$
- But 'Paul is bald and Paul is not bald' is a contradiction!
- Shouldn't [Paul is bald and Paul is not bald] $= 0$?

Higher-Order Vagueness (Again)

- Degrees of truth seem to be a great way of denying that there is a sharp cut off line between being bald and being not bald
  - As people get more hairs, it becomes less and less true to say that they are bald
- But there is still a sharp cut off between people who are completely bald (bald to degree 1), and everyone else
- And there is also still a sharp cut off between people who are completely not bald (bald to degree 0), and everyone else
- But *bald to degree 1* and *bald to degree 0* seem just as vague as *bald*

References

- Edgington, D (1996) *Vagueness by Degrees* in Keefe et al 1996
- Keefe, R et al eds (1996) *Vagueness: A Reader* (Cambridge, MA: MIT Press)
- Sainsbury, RM and Williamson, T (1997) 'Sorites' in Hale and Wright (eds) *A Companion to the Philosophy of Language* (Oxford: Blackwell)