Intermediate Logic Spring
Lecture Six

What Do Second-Order Quantifiers Quantify Over?

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What Do Second-Order Quantifiers Quantify Over?

Second-Order Logic

- **Second-order logic** (SOL) is an extension of first-order logic (FOL), which lets you put variables where predicates go
  - **First-order generalisation:** $\exists x(Fx \wedge Gx)$
  - **Second-order generalisation:** $\exists X(Xa \wedge Xb)$
- Second-order quantifiers are governed by introduction and elimination rules that are directly analogous to the rules for the first-order quantifiers
- SOL also includes a *Comprehension* rule

Comprehension

$$\mid \exists \mathcal{X} \forall \chi (\mathcal{X}\chi \leftrightarrow \mathcal{A}(\ldots \chi \ldots \chi \ldots)) \qquad \text{Comp}$$

- $\mathcal{X}$ must not occur in $\mathcal{A}(\ldots \chi \ldots \chi \ldots)$

Examples

$$\exists X \forall x (Xx \leftrightarrow (Fx \lor Gx))$$

$$\exists X \forall y (Xy \leftrightarrow \exists x Ryx)$$

$$\exists Y \forall y (Yy \leftrightarrow \exists X \exists x (Xx \lor Xy))$$

The Standard Semantics

- Let $\mathcal{F}$ be a new one-place predicate added to the language
- $\forall \mathcal{X}\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$ is true in an interpretation iff $\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$ is true in *every* interpretation that extends the original interpretation by assigning a subset of the domain to $\mathcal{F}$ (without changing the interpretation in any other way)
- $\exists \mathcal{X}\mathcal{A}(\ldots\mathcal{X}\ldots\mathcal{X}\ldots)$ is true in an interpretation iff $\mathcal{A}(\ldots\mathcal{F}\ldots\mathcal{F}\ldots)$ is true in *some* interpretation that extends the original interpretation by assigning a subset of the domain to $\mathcal{F}$ (without changing the interpretation in any other way)

What Do Second-Order Quantifiers Quantify Over?

Quine versus SOL

- This week, we will be looking at Quine's attack on SOL

  (W.V.O. Quine, *Philosophy of Logic*, pp. 64–8)

- According to Quine, second-order logic **isn't really logic!**
- Really, it's a substantial branch of mathematics, dressed up as if it were nothing but logic
- **Quine's pithy slogan:** SOL is "set theory in sheep's clothing"

W.V.O. Quine

To Be Is To Be The Value Of A Variable

- **Quine's criterion of ontological commitment**
  - A theory is *ontologically committed* to the entities it quantifies over

- **Examples:**
  - Modern physics is ontologically committed to electrons, because it contains sentences like '**There are electrons** which...'
  - Number theory is ontologically committed to numbers, because it contains sentences like '**There are numbers** which...'
  - Lewis' extreme modal realism is ontologically committed to possible worlds, because it contains sentences like '**There are possible worlds** which...'

Quine's Question

What do second-order quantifiers quantify over?

Answer 1: Properties

- When I introduced you to SOL last week, I started off saying that second-order quantifiers quantify over **properties**
  - $\exists X(Xa \wedge Xb)$ $\Rightarrow$ there is a property which *a* and *b* both have
  - $\forall X(Xa \rightarrow Xb)$ $\Rightarrow$ *b* has every property that *a* has

- Quine doesn't like this answer, because properties lack clear **criteria of identity**
  - Is the property *triangularity* identical to the property *trilaterality*?

- **A Quinean Slogan:** No entity without identity!
  - If properties lack clear criteria of identity, then they cannot be real entities

Answer 2: Sets

- Alternatively, we might suggest that second-order quantifiers quantify over **sets**
  - $\exists X(Xa \wedge Xb) \Rightarrow$ there is some set which *a* and *b* are both members of
  - $\forall X(Xa \rightarrow Xb) \Rightarrow$ *b* is a member of every set that *a* is a member of
- For Quine, this is a much better answer, because sets *do* have clear criteria of identity
  - Set $a =$ set b iff $\forall x(x \in a \leftrightarrow x \in b)$
- But according to Quine, if second-order quantifiers quantify over sets, then SOL isn't really **logic**

What Do Second-Order Quantifiers Quantify Over?

Logicism

- **Logicism** is the thesis that mathematics is reducible to logic
- Frege was the first great logicist
- He wanted to refute Kant's claim that arithmetic was *a priori synthetic* by showing that all of arithmetic be deduced from pure logic, and was thus **analytically true**

Gottlob Frege

Set Theory and Logicism

- Frege tried a couple of different ways of reducing mathematics to logic, but in the end he decided that the key was identifying numbers with sets
  - $1 =$ the set of all sets with exactly one member
    $$\{x : \exists y \forall z (z \in x \leftrightarrow z = y)\}$$
  - $2 =$ the set of all sets with exactly two members
    $$\{x : \exists y \exists z (\neg y = z \wedge \forall w (w \in x \leftrightarrow (w = y \vee w = z)))\}$$
  - ...
- It was clear that Frege thought that set theory was a branch of **logic**, not mathematics
  - If set theory were not a branch of logic, then identifying numbers with sets would not give us a way of reducing mathematics to logic
- For Frege, sets were special **logical objects**

Naïve Comprehension

- Frege used an axiom he called Basic Law V to govern the behaviour of sets
- The modern descendent of this axiom is **Naïve Comprehension**

(NC) $\exists x \forall y (y \in x \leftrightarrow \mathcal{A}y)$

- What (NC) tells you is that you can use any formula you like to define a set
  - The set defined by $\mathcal{A}$ is $\{x : \mathcal{A}x\}$, i.e. the set of all and only those things which satisfy condition $\mathcal{A}$
- It is not too hard to see why you might have thought of (NC) as a logical law
  - You get a set by *abstracting* on a predicate

Russell's Paradox

- (NC) is revealed to be inconsistent by **Russell's Paradox**
- Here is an instance of (NC):

  $$\exists x \forall y (y \in x \leftrightarrow \neg y \in y)$$

- In other words, there is a set of all sets which are not members of themselves
- **Question:** Is this set a member of itself?
- **Answer:** It is if and only if it isn't!

Russell's Paradox

1	$\exists x \forall y (y \in x \leftrightarrow \neg y \in y)$	
2	$\quad \forall y (y \in a \leftrightarrow \neg y \in y)$	
3	$\quad a \in a \leftrightarrow \neg a \in a$	$\forall_1$E, 2
4	$\quad\quad a \in a$	
5	$\quad\quad \neg a \in a$	$\leftrightarrow$E, 3, 4
6	$\quad\quad \bot$	$\bot$I, 4, 5
7	$\quad \neg a \in a$	$\neg$I, 4–6
8	$\quad a \in a$	$\leftrightarrow$E, 3, 7
9	$\quad \bot$	$\bot$I, 8, 7
10	$\bot$	$\exists_1$E, 1, 2–9

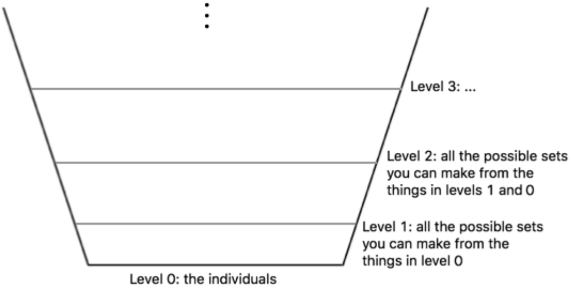
Sets of Sets

- There have been lots of suggested solutions to Russell's Paradox, but I am going to focus on the one that is generally accepted today
- The first thing we should note is that Russell's Paradox would have never got going if you were not allowed to make sets **of sets**
- If you were only allowed to take sets of ordinary things like you and me, then there would be no paradox
- That might make you tempted to deal with the paradox by simply banning us from taking sets of sets
- Unfortunately, that would rob set theory of all of its mathematical value!

The Iterative Conception

- So what we need to do is find a way of introducing sets of sets in a careful, controlled way
- That is just what the **iterative conception of sets** does
  - We start with all the ordinary things (the non-sets, called **individuals**)
  - We then make all the sets we can out of those individuals
  - Then we make all the sets we can out of those sets and our initial individuals
  - And then we make all the sets we can out of those sets of sets, the original sets, and the individuals
  - And we keep doing that forever

The Iterative Conception



- We build up a hierarchy of sets, called the **iterative hierarchy**

Set Theory is not a Branch of Logic

- Mathematicians generally agree that this iterative hierarchy is best described by a set theory known as ZF(C)
- The axioms of this theory are a little complex, but fortunately, we do not need to go through them right now
- The important point is just that the iterative conception of set does not look **purely logical**
  - You do not get sets just by performing a logical operation on a formula
  - You start just by positing some sets
  - You get more sets by "building" them out of other sets!
  - This "building" process generates a complex structure, about which we can ask tough questions (how high? how wide?)

So Second-Order Logic is not a Branch of Logic!

- Earlier we saw Quine's argument that the quantifiers in SOL must quantify over sets
- And we have also seen that to modern eyes, set theory doesn't look like a branch of logic
- So, SOL is not really a branch of logic
- Really, SOL is nothing but set theory, a heavy duty branch of mathematics, in disguise!

In Quine's Words

The values of [second-order] variables are in effect sets; and this way of presenting set theory gives it a deceptive resemblance to logic. One is apt to feel that no abrupt addition to the ordinary logic of quantification has been made; just some more quantifiers, governing predicate letters already present.

In Quine's Words

In order to appreciate how deceptive this line can be, consider the hypothesis '$\exists y \forall x(x \in y \leftrightarrow Fx)$'. It assumes a set $\{x : Fx\}$ determined by an open sentence in the role of 'Fx'. This is the central hypothesis of set theory, and one that has to be restrained in on way or another to avoid the paradoxes.

In Quine's Words

This hypothesis itself falls dangerously out of sight in the so-called [second-order] predicate calculus. It becomes '$\exists Y \forall x(Yx \leftrightarrow Fx)$', and thus evidently follows from the genuine logical triviality '$\forall x(Fx \leftrightarrow Fx)$' by an elementary logical inference. Set theory's staggering existential assumptions are cunningly hidden now in the tacit shift from schematic predicate letter to quantifiable set variable.

(Quine, Philosophy of Logic, pp. 68)

Two Comprehension Principles

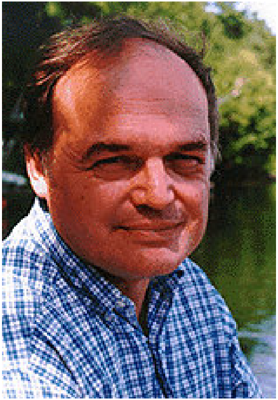
- **Second-Order Comprehension:** $\exists X \forall y (Xy \leftrightarrow \mathcal{A}y)$
- **Naïve Comprehension:** $\exists x \forall y (y \in x \leftrightarrow \mathcal{A}y)$

- You cannot derive Russell's Paradox from Second-Order Comprehension
  - I will leave it to interested students to try to figure out why — e-mail me if you think you've cracked it!
- Nonetheless, if second-order quantifiers quantify over sets, then Second-Order Comprehension is a close relative of Naïve Comprehension
- Quine thinks that no such principle should count as a law of logic

What Do Second-Order Quantifiers Quantify Over?

Boolos versus Quine

- Fans of SOL have offered a number of different responses to Quine
  - I have included a fairly wide range of options in the Reading List
- In this lecture, we will look at the response Boolos gave in 'To Be is to be a Value of a Variable (or to be Some Values of Some Variables)'
- According to Boolos, second-order quantifiers are not set-quantifiers; they are **plural quantifiers**



George Boolos

Two Types of Quantification

- **Singular Quantification**
  - There **is** a minster in York; there **is** no airport in York; some**one is** late for the seminar

- **Plural Quantification**
  - There **are** churches in York; there **are** no mountains in York; some **people are** early for the seminar

- The first-order quantifiers of FOL are primarily designed to symbolise singular quantification

- **Question:** Can we use the first-order quantifiers to symbolise plural quantification?

First-Orderising Plural Quantification

- You can definitely use first-order quantifiers to symbolise **some** plural quantifications
  - There are at least two dogs $\Rightarrow \exists x \exists y(\neg x = y \wedge Dx \wedge Dy)$
- But other cases are trickier
  - Some critics admire only one another
- In order to "first-orderise" this plural quantification, you would need to help yourself to something like set theory
  - $\exists x \forall y(y \in x \rightarrow (Cy \wedge \forall z(Ayz \rightarrow z \in x)))$

Against the Set Theoretic Strategy

- Even if we could always first-orderise plural quantification in this way, it can seem fairly unnatural

- If I eat some Cheerios, I am not eating a set of Cheerios. I am just eating the Cheerios!

- But what is more, there are cases where this set theoretic strategy doesn't work

Russell's Paradox Again

- Imagine that after explaining Russell's Paradox, I went on to say:
  - Nonetheless, there are some well behaved sets: they include all the sets which are not members of themselves, but they don't include any others
- Most mathematicians would agree that I said something true, but if we tried to use the set theoretic strategy to first-orderise it, we would get:
  - $\exists x \forall y (y \in x \leftrightarrow \neg y \in y)$
- This is just the inconsistent instance of Naïve Comprehension which leads to Russell's Paradox!

Plural Quantification as *Sui Generis*

- Rather than trying to somehow reduce plural quantification to singular quantification, we should recognise it as its own thing
- When we use **singular** quantifiers, we quantify over the members of the domain **one at a time**
- When we use **plural** quantifiers, we quantify over the members of the domain **many at a time**
  - When I say that there **are** some churches in York, I am not saying that there **is** a set of churches in York
  - I am just saying that some of the **things** in the domain are churches

The Plural Interpretation of Monadic SOL

- If we don't want to collapse the difference between singular and plural quantification, then we need to find different ways of symbolising them in our formal systems

- Singular quantification is already being symbolised by first-order quantifiers, so how should we symbolise plural quantification?

- **Boolos' Proposal:** We should symbolise plural quantification with *second-order quantifiers*!
  - $\exists X \ldots \Rightarrow$ there are some things such that...
  - $\exists X(Xa \land Xb) \Rightarrow$ there are some things which include both a and b
  - $\forall X(Xa \to Xb) \Rightarrow$ there are no things which include a but not b

Boolos' Response to Quine

- **Quine's Question:** What do second-order quantifiers quantify over?
- **Boolo's Answer:** Exactly the same things that first-order quantifiers quantify over!
- The difference between first-order and second-order quantifiers is not a difference of *domain*
- A second-order quantifier in a given theory will have **exactly the same domain** as the first-order quantifier in that theory

Boolos' Response to Quine

- The difference is just about *how* these quantifiers quantify over that domain
- First-order quantifiers are **singular** quantifiers
  - First-order quantifiers quantify over the domain **one member at a time**
- Second-order quantifiers are **plural** quantifiers
  - Second-order quantifiers quantify over the domain **many members at a time**

Seminar 6

- For Seminar 6, please read:
  - W.V.O. Quine, *Philosophy of Logic*, pp. 64–68
  - George Boolos, ‘To Be is to Be a Value of a Variable (or to Be Some Values of Some Variables)’
- A number of study questions have been posted on the VLE

Lecture and Seminar 7

- Next week, we will start looking at Intuitionistic Logic

- **Required Reading**
  - *An Intuitionistic Logic Primer*, §§1–4
  - A.N. Prior, 'The Runabout Inference Ticket'
  - Nuel D. Belnap, 'Tonk, Plonk and Plink'

- Questions will shortly be posted on the VLE