Intermediate Logic Spring

Lecture One

Modal Logic

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Modal Logic

Variations on Classical Logics

- Last term, we studied **Classical Logic**
    - 'Classical Logic' is just what we call the standard logic which most philosophers and logicians are happy to use
- This term, we are going to look at three variations on Classical Logic:
    - Modal Logic
    - Second-Order Logic
    - Intuitionistic Logic

Varieties of Variation

- Modal Logic and Second-Order Logic are **extensions** of Classical Logic
  - They take everything that Classical Logic has to offer, and then add some more
- Intuitionistic Logic is a **restriction** of Classical Logic
  - It rejects certain classical rules of inference

Why Study These Logics?

- Each one of these logics crops up a lot in philosophy, and so studying them now will help you a lot in your future studies
  - This is especially true of Modal Logic and Second-Order Logic, which philosophers help themselves to *all the time*

- But what is more, each one of these logics is interesting in its own right
  - They have interesting **formal** properties
  - They are surrounded by interesting **philosophical** issues

Teaching

- Contact Hours
  - Weekly lectures (Tuesdays, 09:00-10:00)
  - Weekly seminars (Thursdays, see your timetable)
  - Weekly office hours (Tuesdays 11:00–13:00, Philosophy Department Room 122)

- Procedural Requirements
  - Attend all lectures
  - Complete all required reading
  - Attend, and fully participate in, seminars

Logic Primers

- **There is no textbook for this module**
- However, you will be able to find short introductions to each of the logics we are studying on the VLE
  - These introductions are not full-fledged textbooks, but they will be enough to get you up to speed for this module
- If you are particularly interested in the formal properties of any of the logics we study, then you will be able to find references to proper textbooks in the primers

The Reading List

- There is a full Reading List on the VLE site
- Readings marked **Essential** must be read in preparation for this module
  - Some essential readings are labelled ‘Seminar Reading’. You must read these before the relevant seminar
- Readings marked **Recommended** would be good to read to get a fuller understanding of the material
- Readings marked **Background** are usually more advanced texts, and you only need to read them if you really want a deeper understanding

Assessment

- **Summative Assessment**
  - 2,500 word essay
  - Due Monday Week 1, Summer Term
  - Worth 10 credits (50% of the Intermediate Logic module)
  - A list of questions will be posted on the VLE

- **Formative Assessment**
  - 500 word essay
  - E-mail to me (rob.trueman@york.ac.uk) by noon, Monday Week 6
  - Title: *What puzzles me the most is...*
  - You should lay out an issue that has been puzzling you, explain why it has been puzzling you, and then do your best to resolve that puzzle or difficulty

Assessment

- You will **not** be tested on your ability to prove things using any of the logics we are studying

- You will only be tested on your ability to engage with the philosophical issues surrounding those logics

- However, during this module we **will** look at how to prove things and construct counter-interpretations, for two reasons
  (1) Part of the aim of this module is to equip you to understand those philosophers who do use these logics
  (2) In order to understand the philosophical issues surrounding a logic, you need to have some understanding of how the logic actually works

Modal Logic

What is Modal Logic?

- Modal Logic (ML) is the logic of **necessity** and **possibility**
- We use the symbol $\Box$ to express *necessity*
  - You can read $\Box\mathcal{A}$ as *It is necessarily the case that* $\mathcal{A}$
- We use the symbol $\Diamond$ to express *possibility*
  - You can read $\Diamond\mathcal{A}$ as *It is possibly the case that* $\mathcal{A}$

Varieties of Necessity

- There are lots of different kinds of necessity
  - It is **humanly impossible** for me to run at 100mph, but it is not **physically impossible** for me to move that fast
  - It is **physically impossible** for me to run faster than the speed of light, but it is not **logically impossible** for me to move that fast
- Which kind of necessity does ML deal with? *All of them!*
  - We start with a basic set of rules that govern $\Box$ and $\Diamond$
  - We then add more rules to fit whatever kind of necessity we are interested in

From TFL to ML

- The language of ML is an extension of TFL
  - We could have started with FOL, which would have given us Quantified Modal Logic (QML)
  - QML is much more powerful than ML, but it is also much more complicated
- The basic vocabulary of ML is exactly the same as the basic vocabulary of TFL, except it adds the symbols $\Box$ and $\Diamond$
- ML also has exactly the same rules for how to build sentences out of this vocabulary, but with a couple of extra rules for $\Box$ and $\Diamond$

Sentences of ML

(1) Every atom of ML is a sentence of ML

(2) If $\mathcal{A}$ is a sentence of ML, then $\neg\mathcal{A}$ is a sentence of ML

(3) If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \wedge \mathcal{B})$ is a sentence of ML

(4) If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \vee \mathcal{B})$ is a sentence of ML

(5) If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \rightarrow \mathcal{B})$ is a sentence of ML

(6) If $\mathcal{A}$ and $\mathcal{B}$ are sentences of ML, then $(\mathcal{A} \leftrightarrow \mathcal{B})$ is a sentence of ML

(7) If $\mathcal{A}$ is a sentence of ML, then $\Box\mathcal{A}$ is a sentence of ML

(8) If $\mathcal{A}$ is a sentence of ML, then $\Diamond\mathcal{A}$ is a sentence of ML

(9) Nothing else is a sentence of ML

Modal Logic

System K

- We start with a particularly simple modal system called K, in honour of Saul Kripke
- As before, we will use $\vdash$ to express provability, but we will add a subscript 'K' to indicate that we are using system K
  - You can prove $\mathcal{C}$ from $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ *in system* K
  - $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash_{\mathsf{K}} \mathcal{C}$
- K includes all of the natural deduction rules from TFL, and then adds two more basic rules to govern $\Box$

Modal Ponens

m	$\Box(\mathcal{A} \to \mathcal{B})$	
n	$\Box\mathcal{A}$	
	$\Box\mathcal{B}$	MP, m, n

- We will call this *Modal Ponens*, since it is a souped-up modal version of modus ponens

Necessitation

- **The basic idea:** if $\mathcal{A}$ is a theorem, then so is $\Box\mathcal{A}$
  - Remember, to say that $\mathcal{A}$ is a theorem is to say that $\mathcal{A}$ can be proved without relying on any undischarged assumptions
- This basic idea is easy enough to understand, and seems like quite a good rule
  - If you can *prove* $\mathcal{A}$ without relying on *any* assumptions, then surely it must be **necessarily** true!
- However, figuring out how to actually implement the Necessitation Rule in our proof system is a little tricky

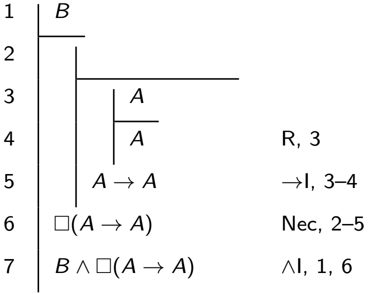
Empty Assumptions

- To implement our Necessitation Rule, we need to introduce a way of showing that a sentence is a theorem *in the middle of a longer proof*
- You are already familiar with the idea that you can trigger a new subproof whenever you like, just by making a new assumption
- We will now push that idea a little further, and say that you can also trigger a subproof by making an ‘empty assumption’

Empty Assumptions

1	B	
2		
3	A	
4	A	R, 3
5	$A \to A$	$\to$I, 3–4
6	$\Box(A \to A)$	Nec, 2–5
7	$B \land \Box(A \to A)$	$\land$I, 1, 6

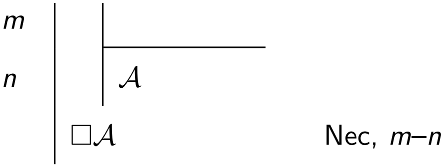
Empty Assumptions



1	B	
2		
3	A	
4	A	R, 3
5	$A \to A$	→I, 3–4
6	$\Box(A \to A)$	Nec, 2–5
7	$B \wedge \Box(A \to A)$	∧I, 1, 6

- When we want to prove that something is a theorem, we start a subproof by making an ‘empty assumption’
- We then write out our proof of this theorem within the subproof

Necessitation: The Official Statement



m

n $\mathcal{A}$

$\Box\mathcal{A}$ Nec, m–n

- No line above line *m* may be cited by any rule within the subproof begun at line *m*.

Some Results

- In system K, you can prove all of the following:

  (1) $\Box(A \wedge B) \vdash_{\mathsf{K}} \Box A \wedge \Box B$

  (2) $\Box A \wedge \Box B \vdash_{\mathsf{K}} \Box(A \wedge B)$

  (3) $\Box A \vee \Box B \vdash_{\mathsf{K}} \Box(A \vee B)$

  (4) $\Box(A \leftrightarrow B) \vdash_{\mathsf{K}} \Box A \leftrightarrow \Box B$

- We will go through some of these as exercises in the seminars, but let's look at how to prove 1 now

$\Box(A \wedge B) \vdash_K \Box A \wedge \Box B$

1	$\Box(A \wedge B)$	
2		
3	$A \wedge B$	
4	A	$\wedge$E, 3
5	$(A \wedge B) \rightarrow A$	$\rightarrow$I, 3–4
6	$\Box((A \wedge B) \rightarrow A)$	Nec, 2–5
7	$\Box A$	MP, 6, 1

1	$\mid\ \Box(A \wedge B)$	
2	$\mid\ \mid$	
3	$\mid\ \mid\ \mid\ A \wedge B$	
4	$\mid\ \mid\ \mid\ A$	$\wedge$E, 3
5	$\mid\ \mid\ (A \wedge B) \rightarrow A$	$\rightarrow$I, 3–4
6	$\mid\ \Box((A \wedge B) \rightarrow A)$	Nec, 2–5
7	$\mid\ \Box A$	MP, 6, 1
8	$\mid\ \mid$	
9	$\mid\ \mid\ \mid\ A \wedge B$	
10	$\mid\ \mid\ \mid\ B$	$\wedge$E, 9
11	$\mid\ \mid\ (A \wedge B) \rightarrow B$	$\rightarrow$I, 9–10
12	$\mid\ \Box((A \wedge B) \rightarrow B)$	Nec, 8–11
13	$\mid\ \Box B$	MP, 12, 1
14	$\mid\ \Box A \wedge \Box B$	$\wedge$I, 7, 13

Modal Logic

- Introduction to Spring
- What is Modal Logic?
- System K
- Possibility
- System T
- S4
- S5

What about Possibility?

- We have now gone over **all** of the basic rules of K
  - K = TFL + MP + Nec
- But you might have noticed that these rules only deal with necessity ($\Box$)
- What happened to *possibility* ($\Diamond$)?

Defining Possibility

- It turns out that we can define possibility in terms of necessity:
    - $\Diamond\mathcal{A} =_{df} \neg\Box\neg\mathcal{A}$
- As a result, we do not really need a special symbol for possibility: we can get by just using $\Box$ and $\neg$
- Still, the system will be much easier to use if we do have a possibility symbol, and so we will add the following definitional rules

Defining Possibility

$$\begin{array}{l|ll} m & \neg\Box\neg\mathcal{A} & \\ & \Diamond\mathcal{A} & \Diamond\text{Def, } m \end{array} \qquad\qquad \begin{array}{l|ll} m & \Diamond\mathcal{A} & \\ & \neg\Box\neg\mathcal{A} & \Diamond\text{Def, } m \end{array}$$

- Importantly, you should not think of these rules as any real addition to K
- They just record the way that $\Diamond$ is defined in terms of $\Box$

Modal Conversion

$$\begin{array}{l|ll} m & \neg\Box\mathcal{A} & \\ & \Diamond\neg\mathcal{A} & \text{MC}, m \end{array}$$

$$\begin{array}{l|ll} m & \Diamond\neg\mathcal{A} & \\ & \neg\Box\mathcal{A} & \text{MC}, m \end{array}$$

$$\begin{array}{l|ll} m & \Box\neg\mathcal{A} & \\ & \neg\Diamond\mathcal{A} & \text{MC}, m \end{array}$$

$$\begin{array}{l|ll} m & \neg\Diamond\mathcal{A} & \\ & \Box\neg\mathcal{A} & \text{MC}, m \end{array}$$

- All of these Modal Conversion rules can be derived from the basic rules of K, plus $\Diamond$Def

Modal Logic

The Limits of K

- K is a very simple system
- It is so simple, that it will not even let you infer $\mathcal{A}$ from $\Box\mathcal{A}$
    - In English: K will not let us infer that $\mathcal{A}$ is **actually** true from the assumption that $\mathcal{A}$ is **necessarily** true!
- Nor will it let us infer $\Diamond\mathcal{A}$ from $\mathcal{A}$
    - In English: K will not let us infer that $\mathcal{A}$ is **possibly** true from the assumption that $\mathcal{A}$ is **actually** true
- This leads us to a new system of ML, T, which we get by adding one new rule to K

The T Rule

$$\begin{array}{r|ll} m & \Box\mathcal{A} & \\ & \mathcal{A} & \text{T, } m \end{array}$$

From True to Possibly-True

- T = K + the T Rule
- Clearly, T allows us to infer $\mathcal{A}$ from $\Box\mathcal{A}$
  - $\Box\mathcal{A} \vdash_{\mathsf{T}} \mathcal{A}$
- But it turns out that it also allows us to infer $\Diamond\mathcal{A}$ from $\mathcal{A}$!
  - $\mathcal{A} \vdash_{\mathsf{T}} \Diamond\mathcal{A}$

Modal Logic

Adding Boxes

- System T allows you to strip away necessity boxes:
  - From $\Box\mathcal{A}$, you can infer $\mathcal{A}$
- But what if you wanted to *add extra* boxes?
  - Can you go from $\Box\mathcal{A}$ to $\Box\Box\mathcal{A}$?
- That would be no problem, **if you had proven $\Box\mathcal{A}$ by applying Necessitation**

$\vdash_T \Box(A \to A)$

1	$\mid\ \mid$	
2	$\mid\ \mid\ \mid A$	
3	$\mid\ \mid\ \mid A$	R, 2
4	$\mid\ \mid A \to A$	→I, 2–3
5	$\mid \Box(A \to A)$	Nec, 1–4

$\vdash_{\mathsf{T}} \Box\Box(A \to A)$

1		
2		
3	A	
4	A	R, 3
5	$A \to A$	→I, 3–4
6	$\Box(A \to A)$	Nec, 2–5
7	$\Box\Box(A \to A)$	Nec, 1–6

But You Can't Always Add an Extra $\Box$ in T

- However, we do not always get $\Box\mathcal{A}$ by applying Necessitation
- It might be, for example, that $\Box\mathcal{A}$ is just an **assumption** that we made
- Are we always free to infer $\Box\Box\mathcal{A}$ from $\Box\mathcal{A}$?
- Not in T we're not, and that seems like a shortcoming of the system
  - It seems intuitive that if $\mathcal{A}$ is necessarily true, then it couldn't have *failed* to be necessarily true
- This leads us to another new system, S4, which we get by adding a new rule to T

The S4 Rule

$$\begin{array}{r|ll} m & \Box\mathcal{A} & \\ & \Box\Box\mathcal{A} & \qquad \text{S4, } m \end{array}$$

Deleting Diamonds

- S4 = T + the S4 Rule
- Clearly, S4 allows us to *add* extra boxes
  - $\Box\mathcal{A} \vdash_{\mathsf{S4}} \Box\Box\mathcal{A}$
- But it also allows us to *delete* extra *diamonds*!
  - $\Diamond\Diamond\mathcal{A} \vdash_{\mathsf{S4}} \Diamond\mathcal{A}$

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Adding Boxes to Diamonds

- In S4, we can always add a box in front of another box
- But S4 does not automatically allow us to add a box in front of a *diamond*
  - S4 does not generally permit the inference from $\Diamond\mathcal{A}$ to $\Box\Diamond\mathcal{A}$
- But again, that might strike you as a shortcoming of S4
  - It seems intuitive that if $\mathcal{A}$ is possibly true, then it couldn't have *failed* to be possibly true
- This leads us to one last system, S5, which we get by adding a different rule to T

The S5 Rule

$$\begin{array}{r|ll} m & \Diamond\mathcal{A} & \\ & \Box\Diamond\mathcal{A} & \quad \text{S5, } m \end{array}$$

Deleting Diamonds Again

- S5 = T + the S5 Rule
- Clearly, S5 allows us to *add* boxes in front of diamonds
  - $\Diamond\mathcal{A} \vdash_{\mathsf{S5}} \Box\Diamond\mathcal{A}$
- But it also allows us to *delete* extra diamonds in front of boxes!
  - $\Diamond\Box\mathcal{A} \vdash_{\mathsf{S5}} \Box\mathcal{A}$

Just One Modal Operator Will Do!

- In S5 you only ever need one modal operator!
  - $\Box\mathcal{A} \vdash_{\mathsf{S5}} \mathcal{A}$
  - $\Diamond\mathcal{A} \vdash_{\mathsf{S5}} \Box\Diamond\mathcal{A}$
  - $\Box\mathcal{A} \vdash_{\mathsf{S5}} \Box\Box\mathcal{A}$

  - $\mathcal{A} \vdash_{\mathsf{S5}} \Diamond\mathcal{A}$
  - $\Diamond\Box\mathcal{A} \vdash_{\mathsf{S5}} \Box\mathcal{A}$
  - $\Diamond\Diamond\mathcal{A} \vdash_{\mathsf{S5}} \Diamond\mathcal{A}$

- So if you have a long string of boxes and diamonds, in any combination whatsoever, you can delete all but the rightmost of them
  - For example: $\Diamond\Box\Diamond\Diamond\Box\Box\Diamond\Box A \dashv\vdash_{\mathsf{S5}} \Box A$

Seminar 1

- The reading for Seminar 1 is:
  - *A Modal Logic Primer*, §§1–3
- Please attempt at least some of the exercises before the seminar. (Why not meet up in groups to do the exercises together?)

Next Week's Lecture and Seminar

- For next week's lecture and seminar, read:
    - *A Modal Logic Primer*, §4
- We will go through some of the exercises in the seminar