Rationality, Morality and Economics

Topic 2, Lecture 1

The Axioms of Expected Utility Theory

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The Axioms of Expected Utility Theory

Decisions Under Risk

- Last week we focussed on decisions under *ignorance*
  - We know what the possible outcomes of the decision would be, but we have no way of assigning any probabilities to those outcomes
- This week, we will look at decisions under *risk*
  - We know what the possible outcomes of the decision would be, and we **can** assign probabilities to these outcomes

An Example

Should you go to see *Glass*?

	Underrated (.2)	Bad as they say (.8)
See it	10	1
Stay home	4	6

- **Principle of Maximising Expected Utility (MEU):** Act so as to maximise your expected utility
  - $EU(A) = \sum_{i=1}^{n}[P(s_i) \times U(A \wedge s_i)]$
- $\text{EU(See it)} = (10 \times 0.2) + (1 \times 0.8) = 2.8$ ×
- $\text{EU(Stay home)} = (4 \times 0.2) + (6 \times 0.8) = 4.8$ ✓

The Axioms of Expected Utility Theory

Where Do Utilities Come From?

	Underrated (.2)	Bad as they say (.8)
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- To apply MEU, your utilities must be measured on an *interval scale*
  - If $u(a) - u(b) > u(c) - u(d)$, then you prefer a to be b more than you prefer c to d
- If I asked, could you really tell me what your utilities are, on an interval scale?

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- If I asked, could you really tell me what your utilities are, on an interval scale?
- If not, where do these utilities come from!?

A Representation Theorem

- **Von Neumann and Morgensten's Representation Theorem**
  - If your preferences satisfy some fundamental axioms, then it is possible to construct an interval utility scale which represents your preferences
- To state these axioms, we use some standard notation:
  - $A \succ B$: you prefer A to B
  - $A \succeq B$: you do not prefer B to A
  - $A \sim B$: you are indifferent between A and B

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  - ApB: lottery with probability p of A, and probability $1 - p$ of B
- **NOTE:** on this set-up, decision problems are choices between lotteries, and lotteries are what you have preferences over

Acts as Lotteries

	Underrated (.2)	Bad as they say (.8)
See it	10	1
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- We can think of seeing the film as a lottery
  - There is a 0.2 chance of seeing an underrated film, and a 0.8 chance of seeing a film that is as bad as people say
- We can similarly think of staying home as a lottery, but with different possible prizes
  - There is a 0.2 chance of missing an underrated film, and a 0.8 chance of missing a bad film

The vNM Axioms

- **vNM1: Completeness**
  - $A \succeq B$ or $B \succeq A$
- **vNM2: Transitivity**
  - If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- **vNM3: Independence**
  - $A \succ B$ if and only if $ApC \succ BpC$ (where $0 < p \leq 1$)
- **vNM4: Continuity**
  - If $A \succ B \succ C$ then there exists some p and q such that $ApC \succ B \succ AqC$ (where $0 < p < 1$ and $0 < q < 1$)

The vNM Representation Theorem

- $\succ$ satisfies vNM 1–4 if and only if there is a function, u, from lotteries to real numbers such that:
  (1) $A \succ B$ if and only if $u(A) > u(B)$
  (2) $u(ApB) = pu(A) + (1 - p)u(B)$
  (3) For any function u' which satisfies (1) and (2), there are some numbers $m > 0$ and c such that: $u'(A) = mu(A) + c$

It's As If You're Maximising Expected Utility

- If your preferences satisfy vNM 1–4, then we can treat you *as if* you were maximising expected utility
  - We can construct an interval utility scale which reflects your preferences
  - Whatever choice you make in a decision problem, that choice will have the highest expected utility on your utility scale
- **IMPORTANT:** You might not think of yourself as maximising expected utility
- Utility scales are the invention of decision theorists, who use them to neatly describe your decision practices

The Axioms of Expected Utility Theory

Where Do Probabilities Come From?

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- In standard expected utility theory, probabilities are subjective
  - Subjective probabilities (aka *credences*) are real numbers which measure your degree of belief
- If I asked, could you really tell me exactly what your subjective probabilities are, or even just to five decimal places?

Where Do Probabilities Come From?

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- If I asked, could you really tell me exactly what your subjective probabilities are, or even just to five decimal places?
- If not, where do these probabilities come from!?

Savage's Representation Theorem

- If your preferences satisfy certain axioms, then there is a utility function u and a probability function p such that:

  (1) $A \succ B$ if and only if $u(A) > u(B)$

  (2) $EU(A) = \sum_{i=1}^{n}[P(s_i) \times U(A \wedge s_i)]$

  (3) For any function u' which satisfies (1), there are some numbers $m > 0$ and c such that: $u'(A) = mu(A) + c$

Savage's Axioms

- Savage shares two axioms with von Neumann and Morgenstern:
  - **Completeness:** $A \succeq B$ or $B \succeq A$
  - **Transitivity:** If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- However, Savage drops the other two axioms, Independence and Continuity, which deal with precise numerical probabilities
- Instead, Savage adds another five axioms, none of which say anything about precise probabilities
- These extra axioms are quite complex, and so we will not go through them now
  - See Box 7.1 of the Peterson's *An Introduction to Decision Theory*

It's As If You're Maximising Expected Utility

- If a person's preferences behave in the right way, we can construct a utility function **and** a probability function for them
- We can then understand their behaviour by thinking of them *as if* they were maximising expected utility
- **IMPORTANT:** You might not think of yourself as maximising expected utility
- Utility scales **and** probability functions are the inventions of decision theorists

An Exciting Thought...

- Philosophers have long thought many (most?) mental states fall into two categories: **beliefs** and **desires**
- Your credence in a proposition represents your degree of belief in that proposition
  - If your credence in the proposition that it will snow tomorrow is 0.2, then you believe that proposition to the degree 0.2
- Your utilities represent your relative desires
  - If your utility for A is 2 and your utility for B is 1, then you desire A more than B
- Savage's Representation Theorem shows that we can reduce your credences (=beliefs) and your desires (=utilities) to your **preferences**

An Exciting Thought...

- Some decision theorists think that we can explain what it means to "prefer" one outcome to another in terms of your **dispositions to choose**
  - To prefer A to B is to be disposed to choose A over B
  - To prefer B to A is to be disposed to choose B over A
  - To be indifferent between A and B is to lack a disposition to choose one over the other
- If this is right, then Savage's Representation Theorem shows that we can reduce your beliefs and desires **to your dispositions to choose**
- This is a version of *behaviourism*, the doctrine that your **mental states** can somehow be reduced to your **behaviour**

The Axioms of Expected Utility Theory

The vNM Axioms

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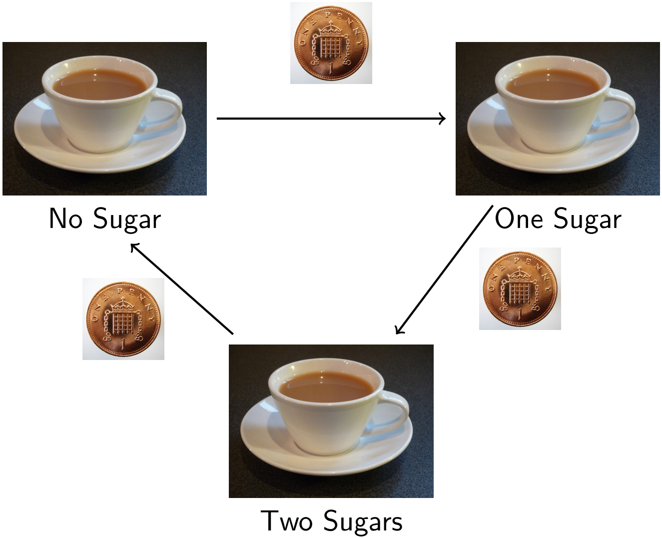
Normative versus Descriptive

- The vNM axioms are not meant to describe real people's real preferences
- They are meant to describe the preferences of an **ideally rational** agent
- We are dealing with **normative** decision theory, not **descriptive** decision theory
  - Descriptive decision theory describes how people *actually* make decisions
  - Normative decision theory describes how people *should* make decisions
- **Big Question:** Are we rationally required to conform to the vNM axioms?

Is Transitivty Rationally Required?

- **Transitivity:** If $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- Sharon prefers tea with two sugars to tea with one sugar, because it tastes nicer, and two sugars isn't that much unhealthier than one
- Sharon also prefers tea with one sugar to tea with no sugar, again because it tastes better, and one sugar isn't that much unhealthier than none
- But Sharon prefers tea with no sugar to tea with two sugars, because two sugars is much unhealthier than none
- Two sugars $\succeq$ one sugar and one sugar $\succeq$ no sugar, but two sugars $\not\succeq$ no sugars
- Are Sharon's preferences irrational?

A Money Pump



Cyclic Preferences can be Money Pumped

- **Cyclic preferences** are preferences which run in a cricle
  - $A_1 \succ A_2 \succ \ldots \succ A_n \succ A_1$
- Someone with cyclic preferences can be **money pumped**
  - They start with A_n
  - Since they prefer A_{n-1} to A_n, they are happy to pay a small fee to swap to A_{n-1}
  - ...
  - Since they prefer A_1 to A_2, they are happy to pay a small fee to swap to A_1
  - Since they prefer A_n to A_1, they are happy to pay a small fee to swap back to A_n
- Many take this to show that cyclic preferences are irrational, but there are objections...

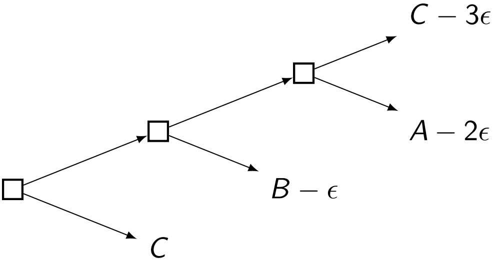
The Axioms of Expected Utility Theory

Objection 1: Other ways of being Intransitive

- If the money pump argument works, it shows that you shouldn't have **cyclic** preferences
  - $A \succ B \succ C \succ A$
- But there are other ways of having intransitive preferences
  - $A \succ B \succ C \sim A$
  - $A \succ B \sim C \sim A$
- We would have to extend the money pump argument somehow to show that intransitive preferences in general are irrational
  - We will look at how we might do that in the next lecture

Objection 2: Backwards Induction

- A clever decision maker could get out of the money pump by applying **backwards induction**



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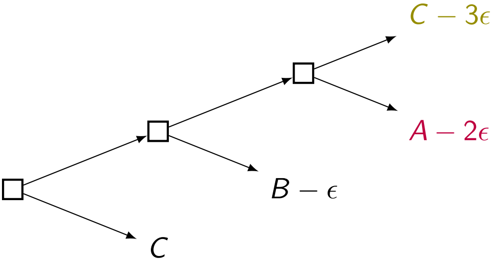

- At the last stage, you would choose $C - 3\epsilon$ over $A - 2\epsilon$

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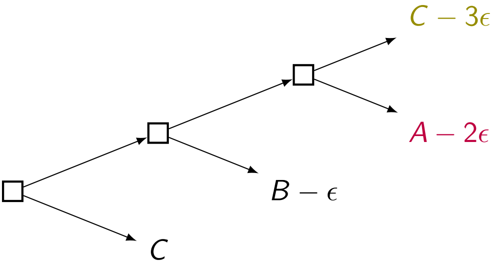

- But in that case, your choice at the second stage is really between $B - \epsilon$ and $C - 3\epsilon$

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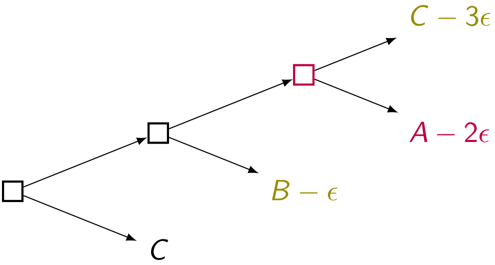

- Since you prefer $B - \epsilon$ to $C - 3\epsilon$, at the second stage you would choose $B - \epsilon$

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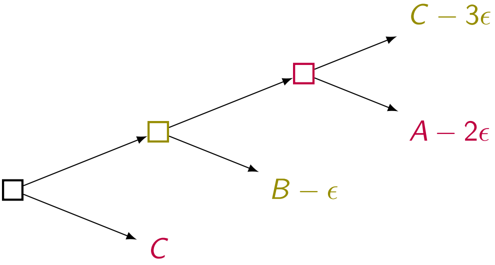


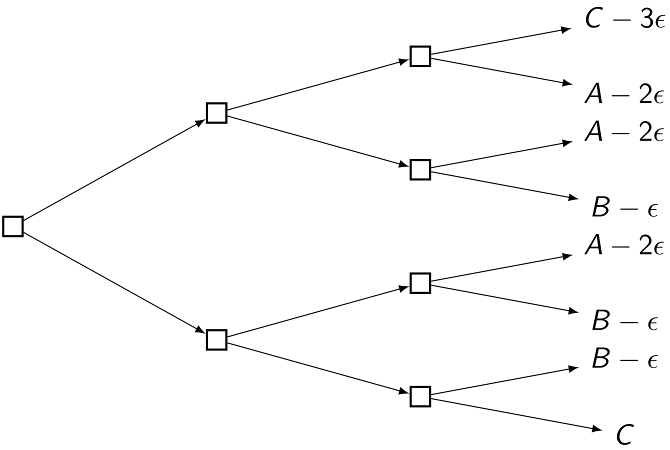
- So your first choice is really just between C and $B - \epsilon$, and so you would exit the pump at $B - \epsilon$

Response: A More Complex Case

- Wlodek Rabinowicz (2000, p. 141) came up with a money pump that will work on someone with cyclical preferences, even if they use backwards induction
- We again assume that we meet someone with cyclical preferences, but this time we make them a complex series of conditional trades

Response: A More Complex Case

$A \succ B \succ C \succ A$



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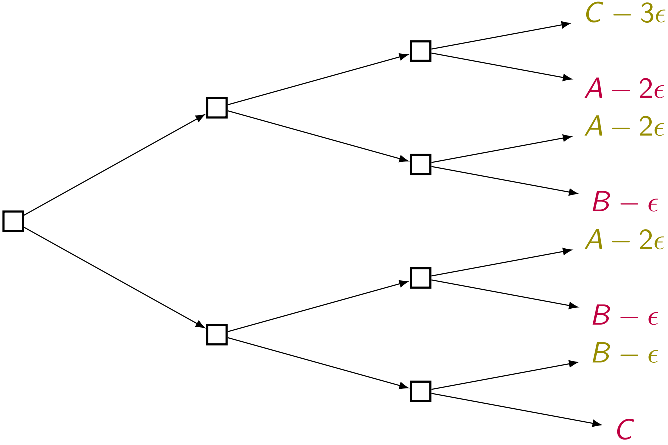

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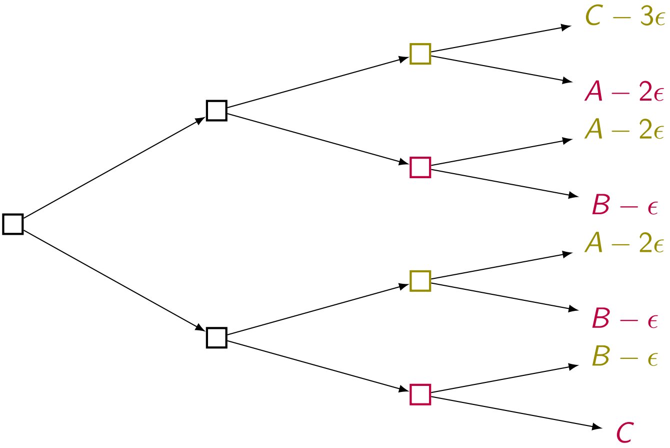

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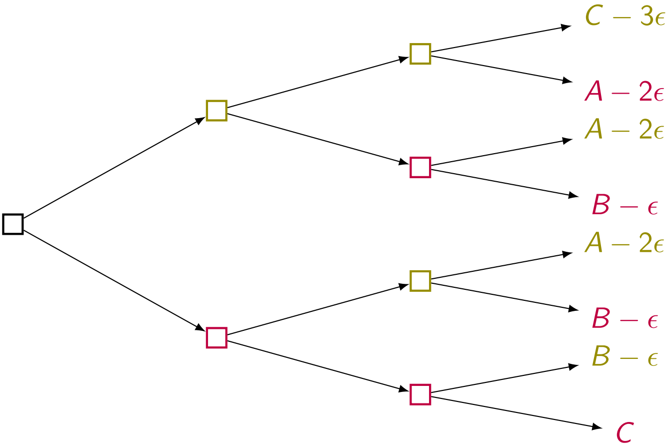

Objection 3: Seeing which way the Wind Blows

- Schick (1986) pointed out that a clever decision maker might notice that she is being money pumped, and having realised that, simply reject one of the offers
  - After we have tricked Sharon into paying 3p just to trade her sugar-free tea for another sugar-free tea, she might notice what is going on
  - At that point, she might just refuse to make another trade!
- But hold on: won't Sharon *inevitably* keep making the trades, since she always prefers the tea we are offering to trade with her?
- Not necessarily...

Objection 3: Seeing which way the Wind Blows

- Schick pointed out that all the money pump arguments rely on a tacit assumption:
  - The value that an agent places on a series of choices is the sum of the values that she places on each choice individually
  - Sharon will always pay 1p to swap sugar-free tea for a tea with one sugar, *no matter what choices she has already made*
- But it is not obvious that agents are rationally required to value series of choices in this way
  - Although Sharon is willing to pay 1p to swap teas in a one-off transaction, she might rationally be unwilling to pay 3p for a sequence of swaps which leaves her with the same cup of tea!
- A similar idea was developed by Rabinowicz (2014), and we will discuss her paper in the seminar

References

- Rabinowicz, Wlodek (2000) 'Money Pump with Foresight', in Michael J. Almeida ed., *Imperceptible Harms and Benefits*, pp. 123–154, Dordrecht: Kluwer
- —— (2014) 'Safeguards of a Disunified Mind', *Inquiry* 57: 356–83
- Schick (1986) 'Dutch Bookies and Money Pumps', *Journal of Philosophy* 83: 112–119