Propositional Functions in Extension

Abstract

In his "The Foundations of Mathematics", Ramsey attempted to marry the Tractarian idea that all logical truths are tautologies and *vice versa*, and the logicism of the *Principia*. In order to complete his project, Ramsey was forced to introduce *propositional functions in extensions* (PFEs): given Ramsey's definitions of 1 and 2, without PFEs even the quantifier-free arithmetical truth that 1≠2 is not a tautology. However, a number of commentators have argued that the notion of PFEs is incoherent. This response was first given by Wittgenstein (1974: 315–7; 1975: 141–3) but has been best developed by Sullivan (1995). While I agree with Wittgenstein and Sullivan's common conclusion, I believe that even the most compelling of Sullivan's arguments is importantly mistaken and that Wittgenstein's remarks are too opaque to be left as the end of the matter. I will, therefore, uncover the fault in Sullivan's argument and present an alternative criticism of PFEs which is Wittgensteinian in spirit without being too mystifying.

1 Introduction

Ramsey's "The Foundations of Mathematics" is best known today for its advocation of a simple rather than ramified theory of types. The broader ambition of the paper, on the other hand, is less well remembered. The Foundations was Ramsey's attempt to marry the following two theses:

(i) All logical truths are tautologies, and *vice versa* (FoM: 155 & 162–3)

(ii) All mathematical truths are logical truths (FoM: 165)

As Ramsey inherited (i) from the *Tractatus* and (ii) from *Principia Mathematica*, a natural name for Ramsey's position is *Tractarian logicism.*[1]

In order to complete his reduction of mathematics to tautologies, Ramsey was forced to introduce *propositional functions in extensions* (PFEs): given Ramsey's definitions of 1 and 2, without PFEs even the quantifier-free arithmetical truth that 1≠2 is not a tautology. However, a number of commentators have argued that the notion of PFEs is incoherent. This response was first given by Wittgenstein (1974: 315–7; 1975: 141–3) but has been best developed by Sullivan (1995). While I

[1]This term is due to Steven Methven.

agree with Wittgenstein and Sullivan's common conclusion, I believe that even the most compelling of Sullivan's arguments is importantly mistaken and that Wittgenstein's remarks are too opaque to be left as the end of the matter. I will, therefore, uncover the fault in Sullivan's argument and present an alternative criticism of PFEs which is Wittgensteinian in spirit without, I hope, being too mystifying.

The interest in this project is partly historical: demonstrating that PFEs are incoherent will show that Ramsey's Tractarian logicism falls apart long before the introduction of the Axioms of Choice and Infinity. Nonetheless, in the conclusion I will try to draw lessons for contemporary philosophy to learn from this period of history.

2 Tractarian Background

This section will serve as a brief account of Ramsey's Tractarian conception of propositions, propositional functions and quantification, which together make up the background against which PFEs are introduced. Following Ramsey (and Wittgenstein) I will call any referring expression a 'name', and the referent of any name an 'object'; like Ramsey, I will use 'individuals' to refer to objects in the more familiar Russellian sense—i.e. the values of the first-order quantifier. Also, for ease I will assume (as Ramsey implicitly does) that our language contains exactly one name for each object and no empty names; if 'a' is a name it will name a, if 'b' is then it will name b, and so on.

We start with propositions.[2] First, an *atomic proposition* is a proposition which could be expressed by a combination of names alone; so, if 'Socrates' and '$\hat{x}$ is wise' are names, then 'Socrates is wise' is an atomic proposition (FoM: 156–7). Second, every proposition is a (perhaps infinite) truth-function of atomic propositions, and every such truth-function is a proposition (FoM: 184–5). Third, the *sense* of a proposition is the class of assignments of truth-values to the atomic propositions on which that proposition is true (FoM: 184). Fourth and finally, there are no distinct propositions with the same sense. (Consequently, there is exactly one proposition true on every assignment of truth-values to the atomic propositions—Tautology—and exactly one true on none—Contradiction.[3])

Next we turn to *propositional functions*,[4] starting with functions of individuals:

[2]By 'propositions' I mean what Ramsey sometimes calls 'proposition types' (FoM: 184).

[3]Following Ramsey (FoM: 161) I will treat Tautology and Contradiction as propositions, albeit the limiting cases.

[4]Again, by 'propositional functions' I mean what Ramsey would describe as types of propositional functions.

> By a propositional function of individuals we mean a symbol of the form '$f(\hat{x}, \hat{y}, \hat{z}, ...)$' which is such that, were the names of any individuals substituted for '$\hat{x}$', '$\hat{y}$', '$\hat{z}$',... in it, the result would always be a proposition. This definition needs to be completed by the explanation that two such symbols are regarded as the same function when the substitution of the same set of names in the one and in the other always gives the same proposition.[5] Thus if '$f(a, b, c)$', '$g(a, b, c)$' are the same proposition for any set of a, b, c, '$f(\hat{x}, \hat{y}, \hat{z})$' and '$g(\hat{x}, \hat{y}, \hat{z})$' are the same function, even if they are quite different to look at. (FoM: 186)

This account is extended to higher-order propositional functions in the obvious way. It is worth emphasising that propositional functions are *symbols* into which other symbols are substituted; the fact that Ramsey describes some propositional functions as being "of individuals" should not mislead us into thinking otherwise.[6] To help keep this in mind, and hence avoid following Sandu (2005: 248) in attributing to Ramsey a confusion between functional expressions and their referents, I will always use single quotation marks when referring to propositional functions.

After introducing us to the general notion of propositional functions, Ramsey singles out a subclass of such functions which he calls 'predicative functions' but we will call 'predicating functions'[7] in order to avoid confusion with the modern use of 'predicative'. Starting with individuals again, an *atomic predicating function of individuals* is "the result of replacing by variables any of the names of individuals in an atomic proposition expressed by using names alone" (FoM: 189);[8] so, if 'Socrates' is a name of an individual and 'Socrates is wise' is an atomic proposition, '$\hat{x}$ is wise' is an atomic predicating function. Next we extend the notion of truth-functions to propositional functions by saying that '$F(\hat{x}_0, \hat{x}_1...\hat{x}_n)$' is a particular truth-function of some propositional functions and propositions iff any value of '$F(\hat{x}_0, \hat{x}_1...\hat{x}_n)$' is that truth-function of the corresponding values of those propositional functions and of those propositions (FoM: 189); so, '$F\hat{x} \vee G\hat{x}$' is a disjunction of '$F\hat{x}$' and '$G\hat{x}$' as for any name 'a', '$Fa \vee Ga$' is a disjunction of 'Fa' and 'Ga'. Finally, every predicating function of individuals is a (perhaps infinite) truth-function of atomic predicating functions of individuals and propositions, and every such truth-function

[5] If we had not made the simplifying assumption that every object has a name, this would have to be re-written as 'two such symbols are regarded as the same function when the substitution of the same set of possible names in the one and in the other always gives the same proposition'.

[6] Nonetheless, we might think certain propositional functions refer to qualities, as Ramsey (1978b: 37) seemed to.

[7] This term comes from (Potter 2002: 210).

[8] In this quotation Ramsey slides between propositions and propositional functions on the one hand, and the symbols which express them, which he called their 'tokens', on the other; however, I think that eliding over the difference is less likely to cause confusion here than explicitly noting it.

is a predicating function of individuals (FoM: 190). Higher-order predicating functions are given a "more or less analogous" treatment (FoM: 190), but we need not be concerned with the details of that treatment here.

With Ramsey's notion of propositional functions to hand, we may give his Tractarian account of quantification. The proposition '$\forall x Fx$' is the conjunction of all the values of '$F\hat{x}$', and the proposition '$\exists x Fx$' is the disjunction of those propositions; similarly, '$\forall\phi\, \phi a$' is the conjunction of every value of '$\hat{\phi} a$', and so on up (FoM: 159). Importantly, this interpretation of the quantifiers does not require that we speak an infinitary language; rather, we use the quantifier notation to finitely express infinite truth-functions of atomic propositions (FoM: 192–3).

I will also take this opportunity to explain Wittgenstein's quantifier convention (*TLP*: 5.53–5.5321),[9] which will prove useful later in this essay. At the basis of the convention is the rule that different names refer to different objects and the corresponding rule for variables. Following Potter (2002: 190), I will use '$\exists x'$' and '$\forall x'$' when Wittgenstein's convention is in force. Ramsey gives a more precise statement of the convention:

> Two different constants must not have the same meaning. An apparent [i.e. bound] variable cannot have the value of any letter occurring in its scope, unless the latter is a variable apparent in that scope. (Ramsey 1991: 159)

So, '$\exists x' Rxa$' is the disjunction of all the values of '$R\hat{x}a$' aside from 'Raa', and '$\forall x' Rxa$' is the conjunction of those values. I should warn that this interpretation of Wittgenstein's quantifiers is not uncontroversial. For present purposes, however, this is not a concern: although I will use these quantifiers, I will at no point be analysing arguments in which Wittgenstein (or anyone else) uses them.

With the required backdrop set, we may now move on to the introduction of PFEs themselves.

3 Propositional Functions in Extension

Following Whitehead and Russell, Ramsey intended to establish (ii) in two steps. First, mathematics is to be reduced to a class theory. This reduction involves defining each natural number (of individuals) n as the class of all n-membered classes of individuals, e.g. 1 is defined as the class of all singletons of individuals and 2 as

[9] See (Potter 2002: 176–7) for an explanation of why Wittgenstein introduced this notation.

the class of all doubletons of individuals. Second, the class theory to which mathematics is reduced is itself to be reduced to logic. This second reduction is to be achieved partly by eliminating class terms in favour of propositional functions (see *Principia*: ∗20); a consequence of this elimination is that every class is defined by a propositional function.

However, if every propositional function is a predicating function then, given (i), there can be no such two-step reduction of mathematics (or even quantifier-free arithmetic) to logic. If 1 is defined as the class of singletons of individuals and 2 as the class of doubletons of individuals, then the mathematical truth that $1 \neq 2$ requires that there be a singleton or a doubleton: otherwise, 1 and 2 would both be empty and hence identical. If, in turn, the existence of a singleton or doubleton of individuals is to be reduced to logic then, assuming (i), it must be a tautology that some propositional function is true of exactly one or exactly two individuals.[10] But, if every propositional function is predicating then this is not a tautology, and this is because predicating functions *do not logically discriminate between individuals*, meaning that it is not contradictory for every individual to satisfy exactly the same predicating functions as every other individual:

Argument. First note that, trivially, no assignment of truth-values to the atomic propositions is contradictory—i.e. Contradiction is not true on any assignment. As atomic predicating functions have atomic propositions as values, it follows that it is not contradictory for every individual to satisfy exactly the same atomic predicating functions as every other individual. Every predicating function is a truth-function of atomic predicating functions and propositions, and therefore it is not contradictory for every individual to satisfy exactly the same predicating functions *simpliciter* as every other individual. *QED.*

The fact that predicating functions do not logically discriminate between individuals entails that if every propositional function is predicating then it is not a tautology that some propositional function is true of exactly one or exactly two individuals: in the consistent worlds where every individual satisfies the same predicating functions as every other individual, every predicating function is either true of every individual or of no individuals.[11]

Moreover, it is worth emphasising that the argument establishing that predicating functions do not logically discriminate between individuals at no point appeals

[10] I.e. '$\exists\phi'\exists x'\forall y'(\phi x \wedge \neg\phi y) \vee \exists\phi'\exists x'\exists y'\forall z'(\phi x \wedge \phi y \wedge \neg\phi z)$' must be Tautology.

[11] I am here assuming that there are more than two individuals. I take it that this assumption is uncontroversial given a background acceptance of the Tractarian account of propositions. Moreover, if there are not more than two individuals, Ramsey's Tractarian logicism is doomed from the start.

to the Tractarian assumption that all necessity is logical necessity. Even if we introduced some non-logical necessity into our Tractarian framework—say to rule out the possibility that there are two individuals who satisfy all the same predicating functions as each other—it would remain the case that no assignment of truth-values to the atomic propositions was *contradictory*, and so it would still not be contradictory for every individual to satisfy exactly the same predicating functions as every other individual.

Ramsey's Tractarian logicism, therefore, requires the introduction of non-predicating functions. Ramsey recommended doing this by "dropping altogether the notion that ϕa says about a what ϕb says about b" and "extensionalising propositional functions completely" (FoM: 203). To this end he introduces his PFEs:

> Such a function of one individual results from any one-many relation in extension between propositions and individuals; that is to say, a correlation, practicable or impracticable, which to every individual associates a unique proposition, the individual being the argument to the function, the proposition its value.
>
> Thus F_e(Socrates) may be Queen Anne is dead,
>
> F_e(Plato) may be Einstein is a great man;
>
> $F_e\hat{x}$ being simply an arbitrary association of propositions $F_e x$ to individuals x. (FoM: 203—trivial notational variant)

Before going further, I would like to pause on a point of detail. Whether PFEs should be thought of as propositional functions properly so called or not is a matter which will be discussed in §§4–6, but it may seem that if we take Ramsey at his word then they could not be: propositional functions yield propositions when completed with *names* (FoM: 186), whereas Ramsey introduces us to PFEs as taking *individuals* as arguments. But in fact there is no tension here. Although propositional functions take names to propositions, Ramsey talks of them taking the referents of those names as arguments (e.g. FoM: 176 & 183). This is unproblematic for Ramsey because the totality of names of individuals is fixed by the totality of individuals (FoM: 187).[12] At any rate, Ramsey made it clear that he meant PFEs to take individuals as arguments in the same way that propositional functions do by his insistence that PFEs *are* propositional functions (e.g. FoM: 203 and his (FoM: 205 fn.1) statement that predicating functions are also PFEs).[13]

[12] Again, if we had not made the simplifying assumption that every object has exactly one name in our language, this claim would have to be re-written as 'the totality of *possible* names of individuals is determined by the totality of individuals'.

[13] This is *contra* Fogelin's (1983: 149–50) claim that PFEs are not at the level of symbols.

If Ramsey's PFEs may be appealed to, then it is a tautology that there is a singleton, doubleton, etc. One way of demonstrating this involves turning first to Ramsey's definition of identity (FoM: 204):

(1) $x = y =_{df} \forall\phi_e(\phi_e x \equiv \phi_e y)$

where 'ϕ_e' may take PFEs as values. When $x = y$, '$x = y$' will be the logical product of '$p \equiv p$', '$q \equiv q$'..., and so will be Tautology. On the other hand, when $x \neq y$, there will be some PFE '$\phi_e\hat{x}$' such that '$\phi_e x$' is 'p' and '$\phi_e y$' is '$\neg p$'. In that case '$x = y$' would be a logical product of propositions including '$p \equiv \neg p$', and so would be Contradiction.

Now take the function '$\hat{x} = a$', where a is an arbitrary individual. If identity were defined by (1) then '$a = a$' would be Tautology, and for any b other than a, '$b = a$' would be Contradiction. Therefore, given that (1) defines a legitimate propositional function, it is a tautology that some propositional function is true of exactly one individual, and hence that there is a singleton. Equally, '$\hat{x} = a \vee \hat{x} = b$' is a function which maps 'a' and 'b' to Tautology and every other name to Contradiction, and so can be used to define a doubleton. By continuing to disjoin such functions we can define ever larger classes, and given the machinery of infinite truth-functions this includes classes of infinite cardinality.

Alternatively, we could use PFEs to define classes directly: any class would be defined by that PFE which maps all of its members to Tautology and the rest to Contradiction. However, we will focus on the definition of classes via identity; for Ramsey the two methods stand or fall together (Potter 2002: 217), and it will often prove easier to assess the adequacy of PFEs in terms of Ramsey's definition of identity.

4 Propositional Functions vs. Functors

If Ramsey (FoM: 204) was right in thinking that his PFEs form an "intelligible notation", then the Tractarian logicist is one step closer to his reduction of mathematics to logic. The remainder of the paper will, therefore, be spent investigating their intelligibility. The first task in this investigation is to ask whether PFEs are best thought of as propositional functions as defined above, or, following a number of commentators,[14] as what I will call *functors*—i.e. symbols which yield complex terms when supplied with names. (More familiar functors include 'the father of $\hat{x}$' and '$\hat{x} + \hat{y}$'.) In the former case, 'F_e(Socrates)' *is the same proposition as* 'Queen

[14] Including Sullivan (1995), Potter (2002: 218–9), Landini (2004: 290) and *perhaps* even Wittgenstein (1974: 317; 1975: 142), although due to his typically enigmatic writings on the subject, I cannot be sure if Wittgenstein is properly included in this group.

Anne is dead', whereas in the latter, 'F_e(Socrates)' *is a complex name referring to* 'Queen Anne is dead'. I mentioned in the previous section that Ramsey thought of PFEs as genuine propositional functions, but in this section I will go further and argue that this is how the Tractarian logicist must think of them.

The first concern which one might have with interpreting PFEs as functors is that doing so reifies propositions: if propositions are the values of functors then by Tractarian lights they must either be objects or, if all terms referring to PFEs disappear on analysis, complexes of objects. But, despite the fact that this is decidedly unTractarian, I will not pursue this worry and will grant that propositions are (complexes of) objects; even once this concession has been made, there are still serious problems for Ramsey on this interpretation.

As Sullivan (1995: 124–5) points out, if PFEs are functors then (1) is not an adequate definition of identity: you cannot construct a sentence by taking a truth-function of names, which is precisely what '$F_e a \equiv F_e b$' would be if PFEs were functors. (Compare 'the father of $a \equiv$ the father of b').[15] At first glance we might think that this problem is easily overcome by replacing (1) with:

(2) $x = y =_{df} \forall \phi_e(\phi_e x \text{ is true} \equiv \phi_e y \text{ is true})$

Indeed, it is an odd fact that although Sullivan (1995: 124–6) clearly thinks that the use-mention blunder which is contained in (1) if PFEs are functors is irreparable, he did not comment on this putative solution. Potter has attempted to address this lacuna by arguing that due to difficulties surrounding the function '$\hat{x}$ is true', this proposal is of no help to the Tractarian logicist:

> We now have the difficulty of explaining in general the relation between a sense p and the proposition 'p is true' [...] A given sense may be expressed by many different symbols (or by none). And the problem is not merely one of selecting, for each sense p, a privileged symbol expressing that sense and deciding by fiat that 'p is true' is to name that symbol, since we wanted 'p is true' to *express* a sense, not to name a symbol, so we should be reduced to using ' "p is true" is true' in an evidently futile regress of attempts to express what we want. (Potter 2002: 218–9)

But as it stands Potter's argument is hard to follow. The remainder of this section will be spent presenting an argument which is similar to Potter's but shows in a more straightforward way that if PFEs are functors then the Tractarian logicist

[15] This objection obviously relies on rejecting the Fregean doctrine that sentences are a species of name. As is well known, such a rejection is built into the Tractarian account of propositions (*TLP*: 3.143 & 4.063). For an interesting discussion of Frege's mistake, see (Sullivan 1994).

cannot escape the use-mention confusion.

Assume that PFEs are functors. If (2) is to make it a tautology that there is a singleton (or doubleton etc.) of individuals as the Tractarian logicist requires, then for some distinct individuals a and b there must be some PFE '$C_e\hat{x}$' such that '$C_e a$ is true $\equiv C_e b$ is true' is Contradiction. But if '$C_e\hat{x}$ is true' is a (complex) predicating function of individuals, then '$C_e a$ is true $\equiv C_e b$ is true' is not Contradiction: predicating functions do not logically discriminate between individuals, and so it is consistent for a to satisfy exactly the same predicating functions as b. Therefore, the Tractarian logicist must insist that '$C_e\hat{x}$ is true' is a non-predicating function; however, as we are thinking of non-predicating functions as functors it follows that the Tractarian logicist must say that, despite appearances, '$C_e a$ is true' and '$C_e b$ is true' are not propositions but complex names, and so even (2) is an ill-formed definition of identity. Of course, we could try replacing (2) with,

(3) $x = y =_{df} \forall\phi_e((\phi_e x \text{ is true}) \text{ is true} \equiv (\phi_e y \text{ is true}) \text{ is true})$

but we have begun "an evidently futile regress of attempts to express what we want".

The Tractarian logicist might be tempted to object that this argument goes wrong in treating '$C_e\hat{x}$ is true' as a complex propositional function of individuals; perhaps we should no more think that '$C_e a$ is true' contains a name of a than we think ' "a is wise" is true' does.[16] But the Tractarian logicist introduced PFEs in an attempt to get around the fact that predicating functions do not logically discriminate between individuals, and PFEs obviously cannot serve this purpose if they are not really functions of individuals.

So if PFEs are functors, (1) cannot be repaired by appealing to truth. Moreover, the argument which established this at no point turned on any peculiar features of truth, and so is wholly general: we could replace '$\hat{x}$ is true' in (2) with any function we liked.[17] So, if PFEs are functors (1) is irreparable. The Tractarian logicist needs PFEs to be genuine propositional functions, not mere functors to propositions.

5 Sullivan on Containment

We have seen that the Tractarian logicist requires PFEs to be propositional functions properly so-called; however, Sullivan (1995: 117–8) presented the following argument to show that they are not. For every atomic predicating function there is a *corresponding* PFE, i.e. a PFE which maps the same names to the same propositions. Therefore, by the identity criterion for propositional functions, if PFEs

[16] Thanks to Tim Button for raising this objection.

[17] This is an important respect in which my argument and Potter's differ; Potter (2002: 221) makes it clear that his argument is meant to rely on the particular nature of truth.

are propositional functions then every atomic predicating function is identical to a PFE. However, atomic predicating functions are "contained" in their values in a sense in which PFEs are not. Sullivan (1995: 118) explains his use of this containment metaphor by saying that whereas the identity of a PFE '$F_e\hat{x}$' is derived from *all* of its values, that of an atomic predicating function '$F\hat{x}$' is derivable from *any* of its values: if you know that 'F(Socrates)' is 'Socrates is wise' then you know that 'F(Plato)' is 'Plato is wise', but you have no way of inferring which proposition 'F_e(Plato)' is from the knowledge that 'F_e(Socrates)' is 'Queen Anne is dead'. Therefore no atomic predicating function is a PFE, and hence PFEs are not propositional functions.[18]

Although I do think there is an important sense in which predicating functions are "contained" in their values and PFEs are not, it is not Sullivan's. Therefore, in this section I will explain what is wrong with Sullivan's elimination of the containment metaphor, and in the next section I will provide a better one.

To begin with, it is tempting to respond to Sullivan by pointing out that there are some atomic predicating functions whose identities cannot be derived from just *any* of their values: as Sullivan himself mentions in his (1992: 97), if you know only that '$F\hat{x}$' is an atomic predicating function such that 'F(Socrates)' is 'Socrates killed Socrates', then you cannot tell whether 'F(Plato)' is 'Plato killed Socrates', 'Socrates killed Plato' or 'Plato killed Plato'. However, there is an easy reply to this objection available to Sullivan. Although he should concede that claiming that the identity of any atomic predicating function can be derived from any of its values oversteps the mark, it remains the case that the identity of an atomic predicating function is *more easily* derived from its values than the identity of the corresponding PFE: e.g. if you know only that '$F\hat{x}$' is an atomic predicating function such that 'F(Plato)' is 'Plato killed Socrates', then you can deduce that '$F\hat{x}$' is '$\hat{x}$ killed Socrates'.

Instead, I will demonstrate that just as with PFEs, the identities of predicating functions in general can only be inferred from *all* of their values. In doing so I will assume that there are two atomic predicating functions, '$\hat{x}$ is wise' and '$P\hat{x}$', whose values nowhere coincide—i.e. there is no name 'a' such that 'a is wise' is the same proposition as 'Pa'.[19] (Against a background of the Tractarian conception of

[18]Sullivan (1995: 117) also suggested that there is a potentially vicious circularity in identifying atomic predicating functions with PFEs: our grasp of PFEs requires an antecedent grasp of the totality of propositions and, Sullivan claims, our grasp of the totality of propositions in turn requires a grasp of atomic predicating functions. But this cannot be quite right, as Ramsey's account of propositions went via names, not propositional functions. Of course, we might be tempted to identify certain functions with certain names, but all that matters is that we did not need a grasp of atomic predicating functions *as atomic predicating functions* to get a hold on the totality of propositions.

[19]It may seem odd that I have made such a complicated assumption rather than simply assume

propositions, this assumption seems uncontroversial.)

We start the demonstration by introducing the predicating function '$P\hat{x} \vee \exists y'(T(\text{Plato}) \wedge \neg Py)$', where '$T\hat{x}$' is '$P\hat{x} \vee \neg P\hat{x}$'. It is worth pausing on one reviewer's objection that this function cannot be predicating as it is defined in terms of a Wittgensteinian quantifier—'$\exists y'$'—which in some sense has (non)identity built into it, and identity is not a predicating function.[20] But while it can be useful to think of the Wittgensteinian quantifiers as having (non)identity built into them, our official understanding of '$\exists y'(T(\text{Plato}) \wedge \neg Py)$' is as the disjunction of all the values of '$T(\text{Plato}) \wedge \neg P\hat{y}$' aside from '$T(\text{Plato}) \wedge \neg P(\text{Plato})$' (see §2). So, '$P\hat{x} \vee \exists y'(T(\text{Plato}) \wedge \neg Py)$' is a truth-function of atomic predicating functions and propositions, and is therefore a predicating function after all.

Next, we can prove that '$P\hat{x} \vee \exists y'(T(\text{Plato}) \wedge \neg Py)$' maps every name other than 'Plato' to Tautology and does not map 'Plato' to Tautology:

Argument. Take the value of '$P\hat{x} \vee \exists y'(T(\text{Plato}) \wedge \neg Py)$' for any argument other than 'Plato', '$Pa \vee \exists y'(T(\text{Plato}) \wedge \neg Py)$'. By the convention described in §2, '$\exists y'(T(\text{Plato}) \wedge \neg Py)$' is a disjunction of the values of '$(T(\text{Plato}) \wedge \neg P\hat{y})$' for every argument other than 'Plato'; however, as the conjunction of a proposition with Tautology is itself that first proposition, '$\exists y'(T(\text{Plato}) \wedge \neg Py)$' is also the disjunction of the values of '$\neg P\hat{y}$' for every argument other than 'Plato'. One such value is '$\neg Pa$'. Therefore '$Pa \vee \exists y'(T(\text{Plato}) \wedge \neg Py)$' is a disjunction including 'Pa' and '$\neg Pa$' as disjuncts, and so is Tautology.

Now take '$P(\text{Plato}) \vee \exists y'(T(\text{Plato}) \wedge \neg Py)$'. This is the disjunction of '$P(\text{Plato})$' and the values of $\neg P\hat{y}$' for every argument other than 'Plato'. '$P\hat{x}$' is an atomic predicating function, and so '$P(\text{Plato}) \vee \exists y'(T(\text{Plato}) \wedge \neg Py)$' is a disjunction of an atomic proposition and the negations of some other atomic propositions, and so is not Tautology. *QED.*

For ease, let's now abbreviate '$P\hat{x} \vee \exists y'(T\hat{z} \wedge \neg Py)$' as '$M(\hat{x}, \hat{z})$'. With the above result in hand, we can see that there is no way of inferring which proposition '$F(\text{Plato})$' is from the knowledge that '$F\hat{x}$' is a predicating function and that '$F(\text{Socrates})$' is the atomic proposition 'Socrates is wise'. Most obviously, '$F\hat{x}$' could be '$\hat{x}$ is wise', but it could equally be '$\hat{x}$ is wise $\wedge\, M(\hat{x}, \text{Plato})$'. In the latter case,

that '$\hat{x}$' is wise' and '$P\hat{x}$' are distinct atomic predicating functions. It is because on Ramsey's account of predicating functions (see §2), there may be some atomic predicating functions which map some names to the same propositions but others to different ones. For instance, if '$R(\hat{x}, \hat{y})$' is an atomic predicating function then so are '$R(\hat{x}, a)$' and '$R(a, \hat{x})$', and of course these functions map 'a' to the same proposition but every other name to different propositions.

[20] More precisely, as predicating functions do not logically discriminate between individuals, there is no predicating function '$\hat{x}{=}\hat{y}$' such that when $x{=}y$, '$x{=}y$' is Tautology, and otherwise '$x{=}y$' is Contradiction.

'F(Socrates)' would still be the proposition 'Socrates is wise' as 'M(Socrates,Plato)' is Tautology and the conjunction of a proposition with Tautology is itself that first proposition; however, 'M(Plato,Plato)' is not entailed by 'Plato is wise',[21] and so 'Plato is wise' and 'Plato is wise $\wedge\, M$(Plato,Plato)' have different senses and are therefore different propositions.

Moreover, even if we were to find out that 'F(Plato)' is 'Plato is wise $\wedge\, M$(Plato, Plato)', we could not yet say what 'F(Aristotle)' is: '$F\hat{x}$' might be '$\hat{x}$ is wise $\wedge\, M(\hat{x}$,Plato)', but it also might be '$\hat{x}$ is wise $\wedge\, M(\hat{x}$,Plato) $\wedge$ $M(\hat{x}$,Aristotle)'. In general then we cannot say which proposition 'Fa' is by inspecting the values of '$F\hat{x}$' for arguments other than 'a' as we can never dismiss the possibility that '$F\hat{x}$' is a conjunction with '$M(\hat{x}, a)$' as a conjunct.

So, predicating functions in general are not "contained" in their values in Sullivan's sense. Furthermore, it is trivially true that for any atomic predicating function there is a corresponding predicating function. But we should not absurdly conclude that predicating functions in general are not propositional functions. Instead, the lesson to be learnt is that the sense in which an atomic predicating function is "contained" in a proper subset of its values is that *if we know that a propositional function is an atomic predicating function*, then we can deduce which function it is from some proper subset of its values; we can make such a deduction because we know that an instance of that function could be constructed in a way which guarantees a certain relation between its values. A predicating function in general is not "contained" in a proper subset of its values in the sense that *if we know only that a propositional function is a predicating function* then we cannot deduce which function it is from any proper subset of its values; we cannot because we know less about the relations between its values. But this is an intensional difference between atomic predicating functions and predicating functions in general, and therefore does not prevent the former being a subset of the latter.

For exactly the same reason, that PFEs are not "contained" in their values in Sullivan's sense but atomic predicating functions are is by itself no reason to think that atomic predicating functions are not PFEs.[22]

[21] An atomic proposition 'p' entails a proposition of the form '$q_0 \vee \neg q_1 \vee \neg q_2$...' where '$q_0$', '$q_1$', '$q_2$'... are different atomic propositions iff 'p' is 'q_0'. As we assumed that there is no 'a' such that 'a is wise' is 'Pa', it follows that 'Plato is wise' does not entail 'M(Plato,Plato)'.

[22] A reviewer has responded to my argument by suggesting that it is a mistake to (like Ramsey) individuate propositions purely via their sense; instead, we should also take into account the manner in which they are expressed. In that case propositional functions would also be individuated partly by their mode of expression, and a predicating function expressed in a "canonical" manner would presumably be recoverable from any one of its instances. There are a number of things to say to this response, but I will only mention the most important. I presume that the "canonical" expression of an atomic predicating function would consist of names and variables alone; so, if 'is wise' is a name, '$\hat{x}$ is wise' would be a canonically expressed atomic predicating function. PFEs, on the other hand, would presumably have a different form of canonical expression; this is why they would not be contained in their instances. Therefore, if we individuated propositions partly

6 Tables and Names

In this section I will offer a sense of "containment" in which we can infer from the fact that predicating functions are contained in their values but PFEs are not that PFEs are not propositional functions.[23] I will approach this task by asking, In what sense do non-predicating propositional functions "map" names to propositions? It is important to stress that in asking this question I am not expressing typical constructivist worries about arbitrary infinities, and to avoid such a misunderstanding we will suppose that Socrates and Plato are the only individuals. Rather, my worry is about how we are to understand the metaphor of mapping in the claim that '$F_e\hat{x}$' maps 'Socrates' and 'Plato' to propositions.

In the case of predicating functions, the mapping metaphor can be explained in terms of substitution. We will call an expression of an atomic proposition using names alone an *atomic proposition instance*, and an expression of an atomic predicating function using names and variables alone an *atomic function instance*. An atomic predicating function '$F\hat{x}$' "maps" 'Socrates' to 'F(Socrates)' and 'Plato' to 'F(Plato)' in the sense that the result of substituting 'Socrates' for every variable in an atomic instance of '$F\hat{x}$' is an atomic instance of 'F(Socrates)', and the result of replacing those variables with 'Plato' is an atomic instance of 'F(Plato)'. The sense in which non-atomic predicating functions "map" names to propositions is then explained in terms of truth-functions of atomic predicating functions and propositions: if '$F\hat{x}$' and '$G\hat{x}$' are atomic predicating functions then '$F\hat{x} \wedge G\hat{x}$' maps 'Socrates' to '$F(\text{Socrates}) \wedge G(\text{Socrates})$' because it is the conjunction of '$F\hat{x}$' and '$G\hat{x}$', which themselves map 'Socrates' to 'F(Socrates)' and 'G(Socrates)' respectively. No similar explanation will help us understand how '$F_e\hat{x}$' maps names to propositions—'$F_e\hat{x}$' is not a truth-function of propositional functions which could have atomic instances—but is there any alternative explanation?

Before trying to answer this question, there is a point worth making clear. There is no difficulty in understanding how a propositional function could map 'Socrates' to 'Queen Anne is dead'. The predicating function 'Queen Anne is dead $\wedge$ ($\hat{x}$ is wise $\equiv$ $\hat{x}$ is wise)', to name just one, unproblematically does just that; by substituting 'Socrates' into that function we, in a sense, say of Socrates that Queen Anne is dead. Equally, the predicating function 'Einstein is a great man $\wedge$ ($\hat{x}$ is wise $\equiv$ $\hat{x}$ is wise)' maps 'Plato' to 'Einstein is a great man'. The difficulty is introducing

in terms of their mode of expression, there would be no atomic predicating functions and PFEs which mapped the same names to the same propositions. But Sullivan's argument relied on their being such atomic predicating functions and PFEs.

[23]Although the arguments of this section are similar in spirit to those contained in the second half of Sullivan's (1995), in which he develops interpretations of Wittgenstein's responses to PFEs, I believe them to be substantially different in detail; however, I will be content even if it turns out that this section only expresses more clearly what Sullivan was trying to say.

a propositional function which simultaneously maps 'Socrates' to 'Queen Anne is dead' and 'Plato' to 'Einstein is a great man'. If we could appeal to a propositional function, '$\hat{x} = $Socrates', which mapped 'Socrates' to Tautology and 'Plato' to Contradiction, and another, '$\hat{x} = $Plato', which mapped 'Plato' to Tautology and 'Socrates' to Contradiction, then we could construct the function '($\hat{x} = $Socrates $\supset$ Queen Anne is dead) $\wedge$ ($\hat{x} = $Plato $\supset$ Einstein is a great man)' which would map 'Socrates' to 'Queen Anne is dead' and 'Plato' to 'Einstein is a great man'. However, '$\hat{x} = $Socrates' and '$\hat{x} = $Plato' are themselves non-predicating: 'Socrates = Socrates $\equiv$ Plato = Socrates' and 'Plato = Plato $\equiv$ Socrates = Plato' are Contradiction, but as predicating functions do not logically discriminate it is not a contradiction for all and only the predicating functions true of Socrates to be true of Plato. Therefore '$\hat{x} = $Socrates' and '$\hat{x} = $Plato' cannot be appealed to in an informative account of how non-predicating functions map names to propositions.

In an attempt to answer our question, the Tractarian logicist might claim that '$F_e\hat{x}$' maps 'Socrates' to 'Queen Anne is dead' and 'Plato' to 'Einstein is a great man' in virtue of the following definitions:

(4) F_e(Socrates) $=_{df}$ Queen Anne is dead

F_e(Plato) $=_{df}$ Einstein is a great man

However, when considering such a suggestion, Wittgenstein (1974: 316–7) correctly objected that "to say that these [two] definitions determine the function [$F_e\hat{x}$] is either to say nothing, or to say the same as the [two] definitions say. For the signs ['F_e(Socrates)' and 'F_e(Plato)'] are no more function and argument than the words 'Co(rn)', 'Co(al)' and 'Co(lt)' are". In other words, if 'F_e(Socrates)' is defined as a whole to mean that Queen Anne is dead, then it is a mistake to think that the occurrence of 'Socrates' in it has been conferred any meaning of its own, and so a mistake to think that it can be replaced by a variable.[24]

Alternatively, the Tractarian logicist might hope to cash out the mapping metaphor with the stipulations that

(5) F_e(Socrates) $\equiv$ Queen Anne is dead

F_e(Plato) $\equiv$ Einstein is a great man

or that

(6) 'F_e(Socrates)' and 'Queen Anne is dead' have the same sense

'F_e(Plato)' and 'Einstein is a great man' have the same sense

[24]See also (*Gl*: §56), (Sullivan 1995: 140) and (Potter 2002: 220–1).

But these stipulations are no better than (4). (5) does not specify a unique proposition for 'F_e(Socrates)' or 'F_e(Plato)' to express: for all that (5) says, 'F_e(Socrates)' could express any proposition with the same truth-value as 'Queen Anne is dead' and 'F_e(Plato)' could express any proposition with the same truth-value as 'Einstein is a great man'. But setting that aside, both (5) and (6) still fail to confer any functional complexity onto 'F_e(Socrates)' and 'F_e(Plato)'. Of course, (5) and (6) do not rule out the possibility that 'F_e(Socrates)' and 'F_e(Plato)' are functionally complex in the way that saying that they are defined as notational variants of 'Queen Anne is dead' and 'Einstein is a great man' does, but neither set of stipulations deliver any such complexity: if we introduced 'p' as shorthand for 'Queen Anne is dead' then 'p' would have the same sense as 'Queen Anne is dead' despite admitting of no logical complexity. Therefore, if we tried to specify the sense of 'F_e(Socrates)' and 'F_e(Plato)' simply by asserting (5) or (6), then both these sentences would lack functional complexity, and this would because we would not have done enough to give them any.

Instead, in an attempt to force some kind of logical complexity upon 'F_e(Socrates)' and 'F_e(Plato)', we might try to define '$F_e\hat{x}$' with the following table:

	$F_e\hat{x}$
Socrates	Queen Anne is dead
Plato	Einstein is a great man

Couldn't we then say that whenever a name is substituted for '$\hat{x}$' in '$F_e\hat{x}$', the result expresses the same proposition as the sentence paired with that name on this table, that '$\forall x F_e x$' expresses the conjunction of every proposition which is expressed by a sentence on the right-hand column of this table, etc, and wouldn't this deliver onto '$F_e\hat{x}$' the required complexity?

The answer is that we could not introduce such a convention if by 'name' we mean a symbol which refers (or even purports to refer) to an object. It is not in virtue of having a certain shape that a sign names something, but in virtue of the use to which it is put in the language; supposing otherwise is to think that a magical bond links certain shapes to certain objects. But if '$F_e\hat{x}$' is defined by this table and convention, then rather than serving to point to an individual in the world, the string 'Socrates' appears in 'F_e(Socrates)' only to direct us to a line of the table. We could have marked the lines of this table with any signs we liked—numerals, letters or squares of colour—and so that we chose to mark each line of this table with strings which look like names of Socrates and Plato should not mislead us into thinking that they are those names. For comparison, suppose that we drew up the table,

	co$\hat{x}$
rn	corn
al	coal
lt	colt

and instituted the convention that the result of substituting a sign for '$\hat{x}$' in 'co$\hat{x}$' should be read as the sign it is paired with on this table. We would, of course, not have thereby introduced a function which maps the name of an individual 'rn' to 'corn'; and that would obviously still be true even if the string 'rn' did appear as the name of an individual elsewhere (Wittgenstein 1974: 317).[25] But as the F_e-convention and this one differ in no (relevant) respect other than the shapes of the signs which they happen to pair, it follows that the former also cannot be used to introduce a function mapping names of individuals to symbols.

Why, though, is the sense of "name" in which the table above does not pair names to propositions important? It is because this *is* the sense in which predicating functions map "names" to propositions: when we replace '$\hat{x}$' in '$\hat{x}$ is wise' with 'Socrates' we thereby generate 'Socrates is wise', in which 'Socrates' occurs as a *name of* Socrates. PFEs were introduced to get around the fact that predicating functions do not logically discriminate between individuals; but if PFEs are not really functions of individuals, i.e. if the "names" which they map to propositions are not names of individuals, then they obviously are not fit for that purpose.

I do not think it is too much of a stretch to describe the attempt to reduce mathematics to "propositional functions" which do not take genuine names to propositions as a species of formalism. Although the whole symbol 'F_e(Socrates)' has a sense, the subsentential element 'Socrates' stands for nothing and has no meaning at all in the appropriate sense. If mathematics were reduced to such "propositional functions" then it would no longer be applicable to the world. To put things picturesquely, although Ramsey wants to drop altogether the notion that 'ϕa' says about a what 'ϕb' says about b, he must not in the process drop the notion that 'ϕa' says *something* about a, that is that 'a' occurs as a name of a in 'ϕa'.

There are two ways in which the Tractarian logicist might try to avoid this difficulty. First, he might suggest replacing the $F_e\hat{x}$-table above with:

	$F_e\hat{x}$
Socrates	Queen Anne is dead $\wedge$ (Socrates is wise $\equiv$ Socrates is wise)
Plato	Einstein is a great man $\wedge$ (Plato is wise $\equiv$ Plato is wise)

As 'Socrates' does appear as a name of Socrates in 'Queen Anne is dead $\wedge$ (Socrates

[25] See also (Sullivan 1995: 120).

is wise ≡ Socrates is wise)', doesn't 'Socrates' appear as a name of Socrates in 'F_e(Socrates)'? Again the answer is 'no'. 'Socrates' still only appears in 'F_e(Socrates)' to indicate which line of the table we should look-up; that the proposition instance with which 'Socrates' is paired on this table contains a name of Socrates is a distracting irrelevance.

Second, the Tractarian logicist could claim that the fact that 'Socrates' is written by 'Queen Anne is dead' on the table expresses a proposition about Socrates, say that Socrates bears some relation R to 'Queen Anne is dead' alone, just as the fact that 'England' and 'London' are paired on the following table,

The capital of $\hat{x}$	
England	London
France	Paris
...	...

can be read as saying that London is the capital of England. Rather than appealing to the conventions sketched out above, the Tractarian logicist could then define '$F_e\hat{x}$' as equivalent to 'The y such that $R(\hat{x}, y)$ is true', meaning that 'Socrates' does appear as a name of Socrates in 'F_e(Socrates)'. However, this proposal is really just a variant of the one which would have us treat PFEs as functors, and so fails for a variant of the same reason.[26] We have "explained" how '$F_e\hat{x}$' maps names to propositions by saying that we ought to read it as a shorthand for 'The y such that $R(\hat{x}, y)$ is true'. But if '$F_e\hat{x}$' is a non-predicating function and is an abbreviation for 'The y such that $R(\hat{x}, y)$ is true', then this latter function must also be non-predicating. So all we have done is replace the mystery of how one non-predicating function, '$F_e\hat{x}$', maps names to propositions with the mystery of how another non-predicating function, 'The y such that $R(\hat{x}, y)$ is true', does. This could only ever be an illusion of progress.

None of the explanations of how non-predicating functions map names to propositions considered so far have been adequate. Further, we can see that there is no possibility of an adequate explanation: the values of non-predicating propositional functions must be artificially paired with their "arguments", but in doing so we remove the possibility that they map "names" in the appropriate sense to propositions at all. This is the key to understanding the important sense in which predicating functions are "contained" in their values and PFEs are not. If we were to try to introduce the symbol '$G_e\hat{x}$' as the PFE which corresponds to '$\hat{x}$ is wise' on the supposition that the only individuals are Socrates and Plato, we would have to do so

[26] Indeed, in the formal system of *Principia* (∗30), which Ramsey endorsed, a functor (or as Whitehead and Russell would say, a descriptive function) '$f\hat{x}$' is defined shorthand for something of the form 'the y such that $R(\hat{x}, y)$'.

with the following table (or some equivalent):

	$G_e\hat{x}$
Socrates	Socrates is wise
Plato	Plato is wise

along with the convention that whenever a "name" is substituted for '$\hat{x}$' in '$G_e\hat{x}$', the result is to be read as an instance of the proposition paired with that "name" on this table, etc. But we can now see that the string 'Socrates' does not play the same sort of role in 'Socrates is wise' and in 'G_e(Socrates)': in the former case 'Socrates' is *a name of Socrates*, in the latter it isn't. The important sense in which predicating functions are "contained" in their values but PFEs are not is, then, that a PFE requires the external pairing of its arguments and values by the means of a table or a similar device, whereas the values of a predicating function neither require nor admit of such a specification.

I suppose that the Tractarian logicist might try to escape this conclusion by insisting that the sense in which PFEs "map" names to propositions is primitive, and so not to be explained by appealing to something like the above table. But in that case, the Tractarian logicist might as well have simply insisted from the start that there just are propositional functions which logically discriminate between individuals, rather than giving us a pseudo-account of such functions. Moreover, how are we ever to grasp this primitive sense if we cannot explain how a single PFE maps 'Socrates' and 'Plato' to propositions? As Wittgenstein (1974: 317) said, "the rules for [propositional] functions in the old sense of the word don't hold [for PFEs] at all", and until some account of how PFEs "map" names to propositions, PFEs can "only be understood as a rebus in which the signs have some kind of spurious meaning".

7 Conclusion

In the last section we saw that Ramsey cannot coherently introduce non-predicating propositional functions: if 1 is defined as the class of singletons and 2 as the class of doubletons, then such functions are needed to reduce mathematics to tautologies. But, given that Tractarian logicism is not (and has never been) popular, how important is this conclusion? Well, we now better understand why Tractarian logicism fails. Most people think that there is no workable Principian logicism because of an inevitable reliance on the non-logical Axioms of Reducibility, Choice and Infinity. But Tractarian logicism makes no appeal to the Reducibility, and if PFEs could be appealed to then Choice and Infinity would plausibly be tautologies (FoM: §V). Now we understand that Tractarian logicism really fails because PFEs do not form, in Ramsey's words, "an intelligible notation".

More importantly though, our discussion of Tractarian logicism has wider implications for contemporary philosophy. But before saying just what these consequences *are*, it might be useful to say what they *aren't*. It might be tempting to think that if my objections to PFEs are effective then they will be equally effective against any set theory in which the existence of a singleton (or doubleton etc.) is assumed. This is not right: my arguments only show that it is not a Tractarian tautology that there are any singletons, not that there aren't any. Given how entrenched the assumption that there are singletons is in mathematical practice, this is surely a good thing.

What the considerations against the intelligibility of PFEs do tell us is that certain types of implicit definition are illegitimate.[27] An example of such an implicit definition is the attempt to give a sense to '$\hat{x}$ is true' by merely stipulating some of the instances of

'p' is true iff p

To think that such stipulations gives sense to '$\hat{x}$ is true' is to make exactly the mistake of someone who thought that we could give sense to '$F_e\hat{x}$' by stipulating (5) or (6). And it would be no better to try to implicitly define '$\hat{x}$ is true' with the table:

$\hat{x}$ is true	
'Queen Anne is dead'	Queen Anne is dead
'Einstein is a great man'	Einstein is a great man
...	...

If we were to, ' "Queen Anne is dead" ' would not appear as a name of the proposition(/sentence) 'Queen Anne is dead' in ' "Queen Anne is dead" is true', but only as an arrow pointing to a line of the table. To pick up an earlier analogy, we would have a formalist theory of truth. More generally, we cannot introduce a functional expression by simply using the same string of symbols in a number of sentences.[28] Given its ubiquity, understanding the problems with certain kinds of implicit definition should have far reaching consequences.[29]

[27] I am, of course, not trying to impugn the model-theoretic notion of implicit definition: a theory implicitly defines a non-logical term 'T' iff any two models of the theory with the same domain and the same interpretation of all the other primitive terms give the same interpretation of 'T'.

[28] Thanks to Peter Smith for giving me such a simple formulation of my conclusion.

[29] Thanks are due to (in no particular order) Michael Potter, Tim Button, Nathan Wildman, Christina Cameron, Peter Smith, Fraser MacBride and three anonymous reviewers. Most of all, though, I would like to express my gratitude to Steven Methven; at one stage we were going to write this paper together, and although things didn't turn out that way, the paper would have be immeasurably worse if it had not been for his many contributions.

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