The Philosophy of Physics
Lecture Two

Non-Euclidean Geometry

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Non-Euclidean Geometry

Why Look at Geometry?

- Last week we compared Newtonian Spacetime to Galilean Spacetime
- In effect, Newtonian Spacetime is a four-dimensional **Euclidean** geometry
- Galilean Spacetime discards some of the structure of Newtonian Spacetime, and so is technically a kind of **non-Euclidean** geometry
- It turns out that non-Euclidean geometry is essential to much of modern physics!

The Prehistory of Geometry

- Geometry is the theory of points, lines, planes and the relations between them
- Geometry predates the ancient Greeks: the ancient Egyptians, Babylonians and Chinese all had geometrical knowledge
- But before the Greeks, geometry was a collection of rule-of-thumb procedures whose adequacy was to be assessed empirically

Euclid

- The Greeks deductively derived geometry from a set of postulates
- Euclid was not the first to contribute to this project, but he was the most successful
- In his masterpiece, *Elements* (c. 300BCE), Euclid attempted to derive 465 theorems from 5 postulates

Euclid

Definitions and Common Notions

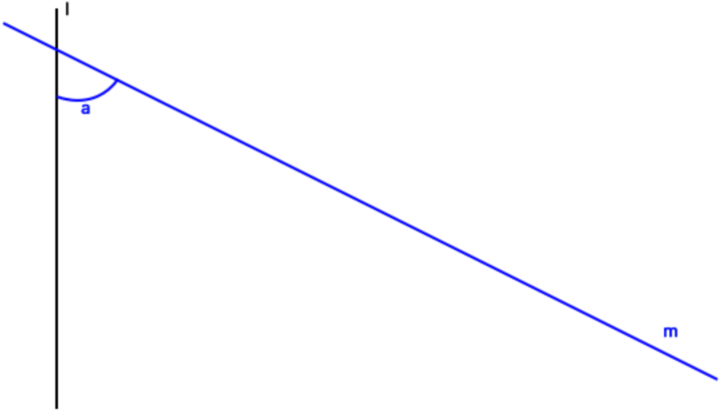
- As well as his 5 postulates, Euclid used 23 definitions
  - A point is that of which there is no part
  - A line is a length without breadth
  - A circle is a figure contained by one line, such that all straight lines falling upon it from one point within the figure equal one another
- He also relied on 5 'common notions'
  - Equals added to equals results in equals
- These common notions seem roughly analogous to what we would call **logical** axioms — they are purely general, not specifically geometrical

The Postulates of Euclidean Geometry

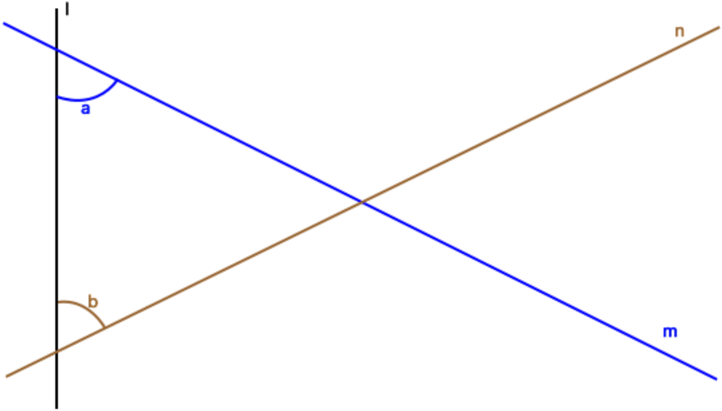
1. Given any two points p and q, exactly one line can be drawn which passes through p and q
2. Any line segment can be indefinitely extended
3. A circle can be drawn with any centre and any radius
4. All right angles are congruent to each other
5. If a line l intersects two distinct lines m and n such that the sum of the interior angles a and b is less than two right angles, then m and n will intersect at some point

The Fifth Postulate

The Fifth Postulate



The Fifth Postulate



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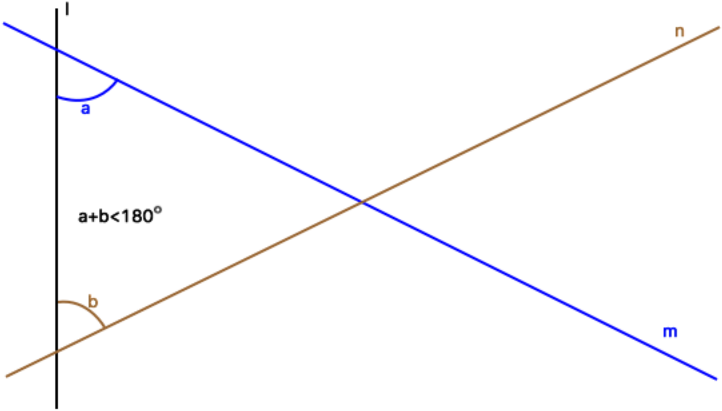

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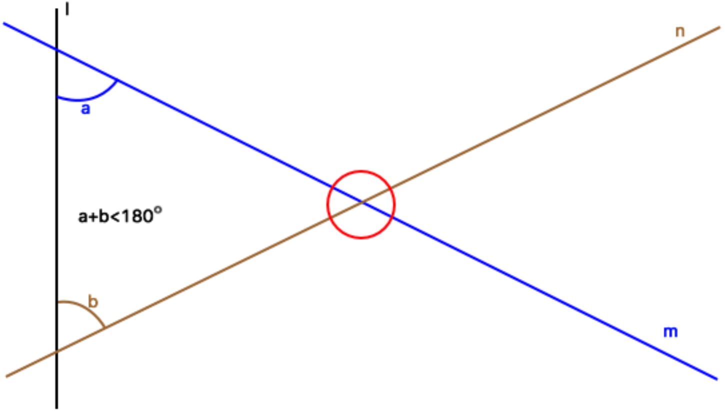


Playfair's Postulate

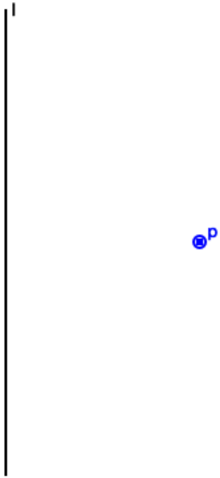
- Postulate 5 was always looked on with suspicion
- This was partly because it was quite complex, but Playfair came up with a simpler version of the postulate:

5′ For every line l and for every point p that does not lie on l, there is exactly one line m that can be drawn through p that is parallel to l

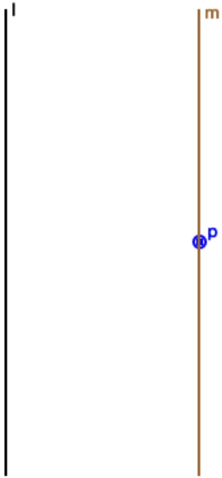
- l and m are parallel iff they do not intersect (and are on the same plane)

Playfair's Postulate

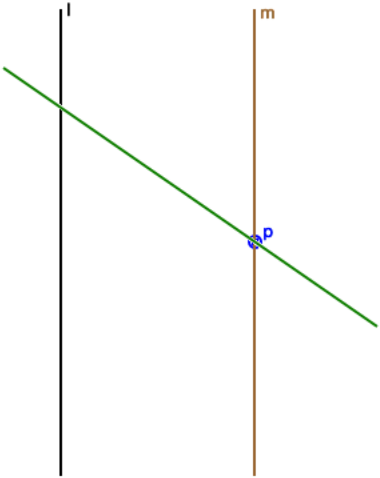
Playfair's Postulate



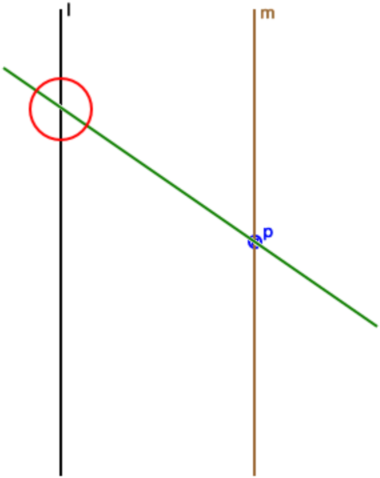
Playfair's Postulate



Playfair's Postulate



Playfair's Postulate



But Still...

- Plenty of mathematicians thought that Postulate 5 looked dodgy
- Postulates 1–4 are, in a sense, abstractions from what we can do with a straight-edge and a compass
- Postulate 5, however, does not look like such an abstraction

But Still...

- Playfair's Postulate tells us that if we have a given line *l* and draw two more lines *m* and *n* which intersect each other at some point, then at least one of *m* and *n* will intersect **somewhere** with *l*
- But we might have to go a **very** long way down the line to find this point of intersect
- In real life, we never really draw lines but *line segments*
- Postulate 5 is not true of the "lines" we draw!

Non-Euclidean Geometry

Abandoning Postulate 5

- As a result of the oddity of Postulate 5, many mathematicians tried to derive it from Postulates 1–4
- Eventually, though, a number of mathematicians started to explore the possibility that postulate 5 was not entailed by postulates 1–4
- They replaced Postulate 5 with:

5H There exists a line l and point p not on l such that at least two distinct lines parallel to l pass through p

- This geometry is called **hyperbolic geometry**

A Brief History of Hyperbolic Geometry

- Gerolamo Saccheri wanted to show that Postulate 5 could be derived from Postulates 1–4

- He published a book in 1733 called *Euclid Vindicated From Every Blemish*, in which he showed that rejecting Postulate 5 leads to all sorts of odd consequences
  - In hyperbolic geometry, for every line, l, and point not on l, p, there are infinitely many lines passing through p which are parallel to l
  - In hyperbolic geometry, no figures can have the same shape but be different sizes
  - In hyperbolic geometry, there are no rectangles

- However, while these sorts of results certainly look odd to Euclidean eyes, none of them are actually self-contradictory

A Brief History of Hyperbolic Geometry

- János Bolyai was the first to publish a treatise on hyperbolic geometry as its own, internally coherent kind of geometry (1831)
- It was as an appendix to a book by his father, Wolfang Bolyai, who had actually spent much of his life trying to derive Postulate 5 from 1–4
- Wolfang Bolyai was so proud of his son's work that he sent it to the greatest mathematician of the day, Carl Gauss, who was also Wolfang's friend
- Gauss didn't react as anyone expected: he claimed that he had already reached all of János's conclusions in unpublished work
- We now know that he wasn't lying!

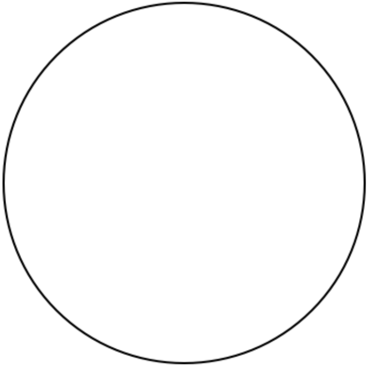
A Brief History of Hyperbolic Geometry

- Nikolai Lobachevsky was actually the first to publish anything on hyperbolic geometry in 1829
- At first his work was little read because it was written in Russian, and the few Russian mathematicians who read it criticised it fiercely
- In 1840, Lobachevsky's work was published in German, and was highly praised by Gauss
- Lobachevsky's work was so influential that hyperbolic geometry is sometimes called **Lobachevskian geometry**

The Consistency of Hyperbolic Geometry

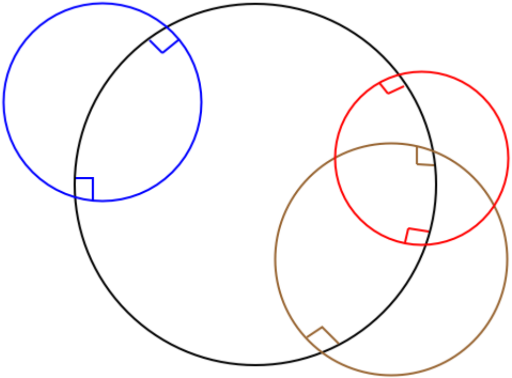
- How do we know that hyperbolic geometry is consistent?
- We can build models of hyperbolic geometry **within** Euclidean geometry
- In other words, we can re-define the words ‘plane’ and ‘line’ so that hyperbolic geometry can be made true within Euclidean geometry

Poincaré's Model



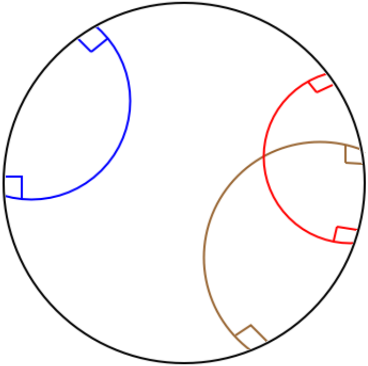
- Consider a 2-dimensional circle on a Euclidean plane

Poincaré’s Model



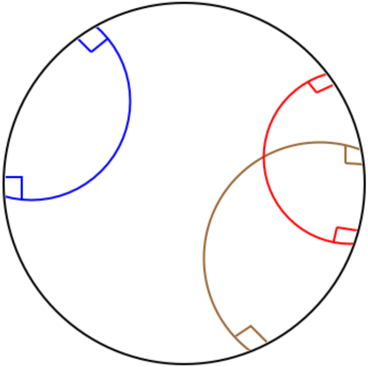
- Add some circles which intersect the first at 90°

Poincaré's Model



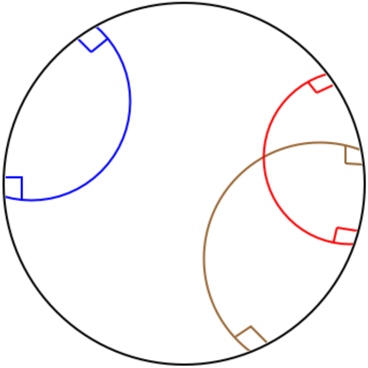
- Now reinterpret the word 'plane' to mean the interior of the black circle

Poincaré's Model



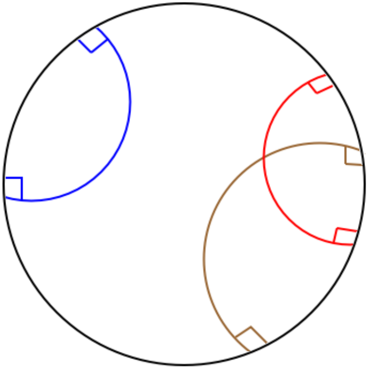
- Reinterpret the 'line' to mean an arc of a circle meeting the black perimeter at 90° (not including end points)

Poincaré's Model



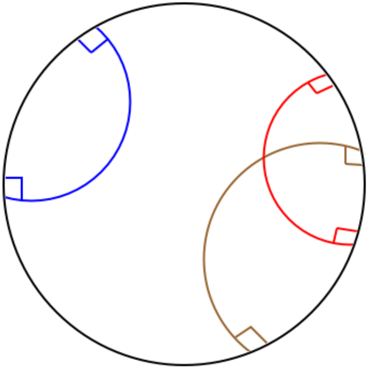
- (We also count all diameters of the black circle — again not including end points — as 'lines')

Poincaré's Model



- All the other geometrical words, like ‘point’, ‘intersect’, ‘parallel’, etc can be left as they are

Poincaré's Model

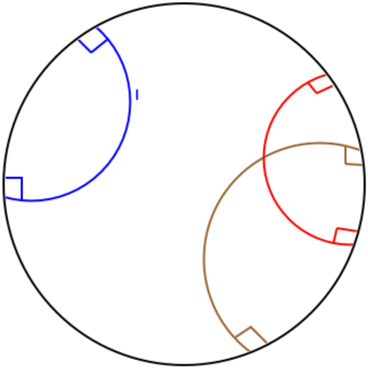


- We now have a model of hyperbolic geometry within Euclidean geometry

Poincaré's Model

5H There exists a line l and point p not on l such that at least two distinct lines parallel to l pass through p

Poincaré's Model



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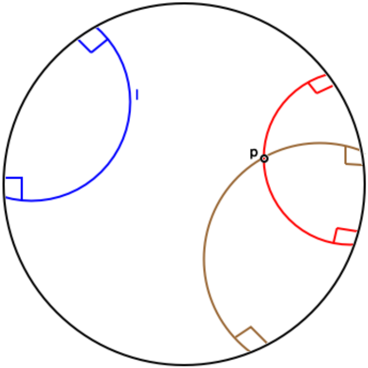
Poincaré's Model



- We now have a model of hyperbolic geometry within Euclidean geometry

Using Poincaré's Model (1)

- We can use this model to show that if Euclidean geometry is consistent, then so is hyperbolic geometry
- It gives us a way of translating theorems of hyperbolic geometry into theorems of Euclidean geometry
- So if the theorems of hyperbolic geometry contradicted each other, then so would the theorems of Euclidean geometry
  - This relies on a background assumption that the translation procedure cannot turn a contradiction into something consistent
  - This is true, since we are translating primitive expressions into complex ones, and that procedure never eliminates contradictions

Relative Consistency

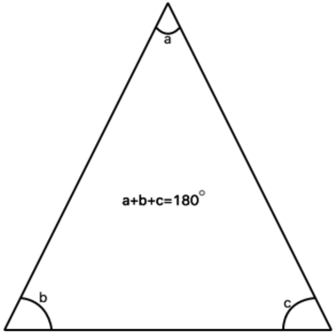
- In modern terminology, we say that hyperbolic geometry is consistent **relative to** Euclidean geometry
- In fact, this is only one relative consistency proof, and we can go much further
- In *Die Grundlagen der Geometrie* (1899), David Hilbert proved that hyperbolic geometry was consistent relative to the arithmetic of real numbers

David Hilbert

Using Poincaré's Model (2)

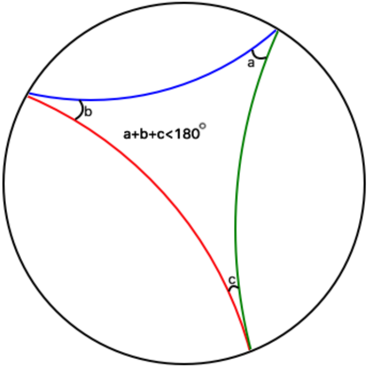
- So one of the things we can use Poincaré's model to do is prove that hyperbolic geometry is consistent relative to Euclidean geometry
- But we can **also** use it to illustrate the theorems of hyperbolic geometry

Using Poincaré's Model (2)



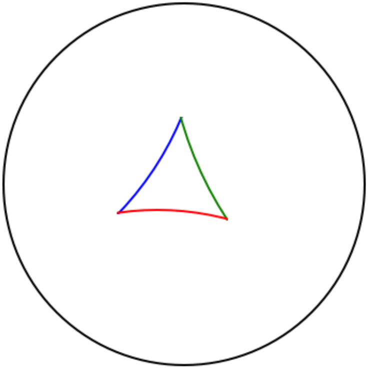
- In Euclidean geometry, the sum of the interior angles of a triangle is always 180°

Using Poincaré’s Model (2)



- But in hyperbolic geometry, the sum of the interior angles of a triangle is always less than 180°

Using Poincaré's Model (2)



- Although as triangles get smaller, their internal angles get closer to 180°

However!

- We should not think that hyperbolic geometry is **really** about a Euclidean circle that is intersected by other Euclidean circles at 90°
- Euclidean geometry is about the Euclidean plane and Euclidean lines
- Hyperbolic geometry is about the hyperbolic plane and hyperbolic lines
- It is just that we can usefully **represent** hyperbolic planes and lines with Euclidean circles

Non-Euclidean Geometry

Another way of Abandoning Postulate 5

- Hyperbolic geometry rejects Postulate 5

  5 For every line l and for every point p that does not lie on l, there is exactly one line m that can be drawn through p that is parallel to l

- And it replaces it with:

  5H There exists a line l and point p not on l such that at least two distinct lines parallel to l pass through p

Another way of Abandoning Postulate 5

- But alternatively, we could reject Postulate 5 and then replace it with:

5E There are no parallel lines

- The best known geometry which has 5E instead 5 is called **elliptic geometry**
  - It is also known as **Reimannian geometry**, but be careful: Reimann came up with a whole range of geometries!

The Consistency of Elliptic Geometry

- If we just add 5E to the other four Postulates, then we actually get a contradiction
  - Postulates 1–4 entail that there are parallel lines all by themselves!

- We can tweak the other Postulates in a number of ways to restore consistency, but in elliptic geometry we replace 2

  2 Any line segment can be indefinitely extended

  with this:

  2E Lines are unbounded

- The reason for making this change will become clear in a moment

The Consistency of Elliptic Geometry

- So these are the postulates of elliptic geometry:

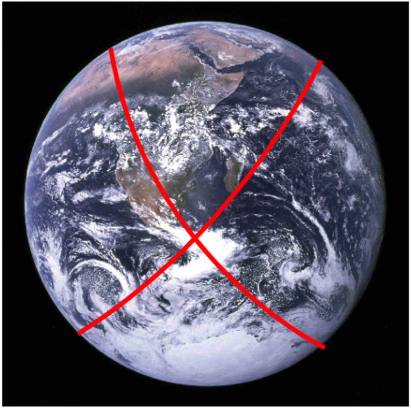
  1 Given any two points p and q, exactly one line can be drawn which passes through p and q
  2E Lines are unbounded
  3 A circle can be drawn with any centre and any radius
  4 All right angles are congruent to each other
  5E There are no parallel lines

- Just like hyperbolic geometry, we can prove that elliptic geometry is consistent relative to Euclidean geometry by building a model of elliptic geometry within Euclidean geometry

The Spherical Model

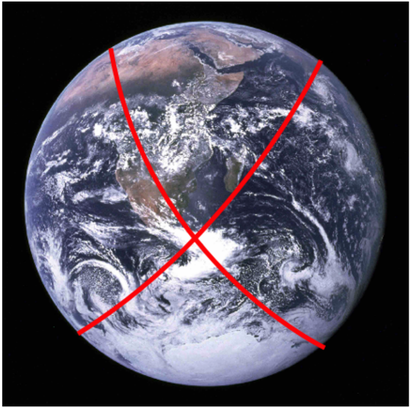
- Reinterpret the word ‘plane’ to mean the surface of a sphere

The Spherical Model



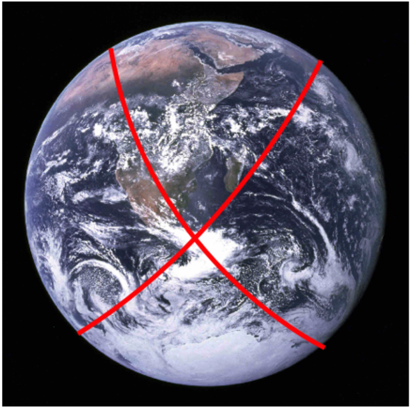
- And reinterpret the word ‘line’ to mean a great circle around the sphere

The Spherical Model



- All the other geometrical words, like ‘point’, ‘intersect’, ‘parallel’, etc can be left as they are

The Spherical Model



- We now have a model of elliptic geometry within Euclidean geometry

Using the Spherical Model (1)

2E Lines are unbounded

- Although every great circle is only finitely long, they are all unbounded, in the sense that you can keep following them without ever reaching an end

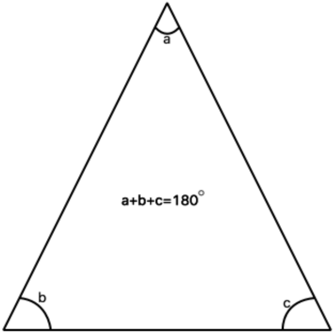
5E There are no parallel lines

- Every great circle intersects every other great circle

Using the Spherical Model (2)

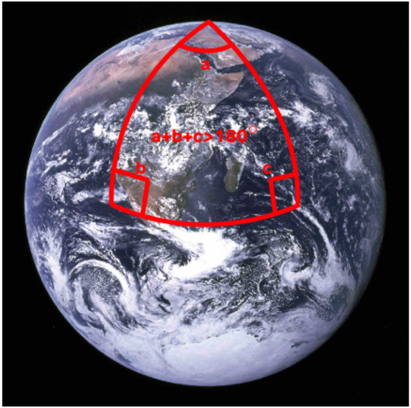
- One of the things that we can use the spherical model to do is prove that elliptic geometry is consistent relative to Euclidean geometry
- But we can also use it to illustrate the theorems of elliptic geometry

Using the Spherical Model (2)



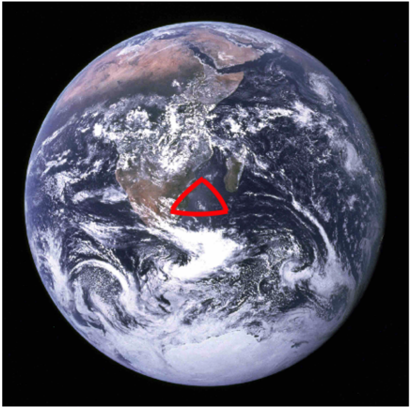
- In Euclidean geometry, the sum of the interior angles of a triangle is always 180°

Using the Spherical Model (2)



- But in elliptic geometry, the sum of the interior angles of a triangle is always greater than 180°

Using the Spherical Model (2)



- Although it gets closer to 180° the smaller the triangle gets

However!

- We should not think that elliptic geometry is **really** about the surface of a Euclidean sphere that is surrounded by various great circles
- Euclidean geometry is about the Euclidean plane and Euclidean lines
- Elliptic geometry is about the elliptic plane and elliptic lines
- It is just that we can usefully **represent** the elliptic plane and elliptic lines with the surface of a Euclidean sphere and Euclidean great circles

Non-Euclidean Geometry

Hyperbolic and Elliptic Geometry

- Elliptic geometry is the opposite, or **dual**, of hyperbolic geometry

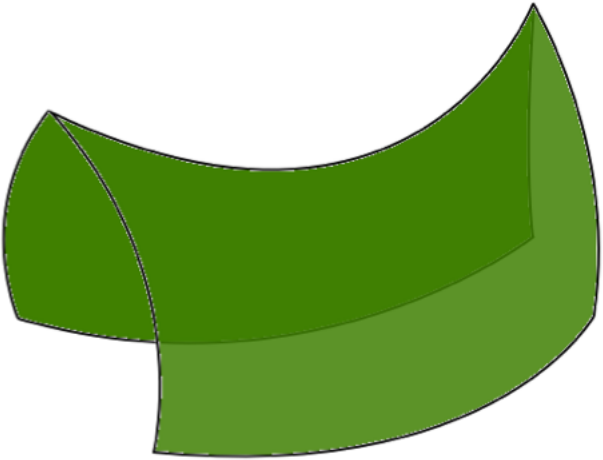
Hyperbolic Geometry	**Elliptic Geometry**
There are infinitely many lines which are parallel to line l and which pass through point p	There are no parallel lines
The interior angles of a triangle sum to less than 180°	The interior angles of a triangle sum to greater than 180°

- This is not a coincidence!

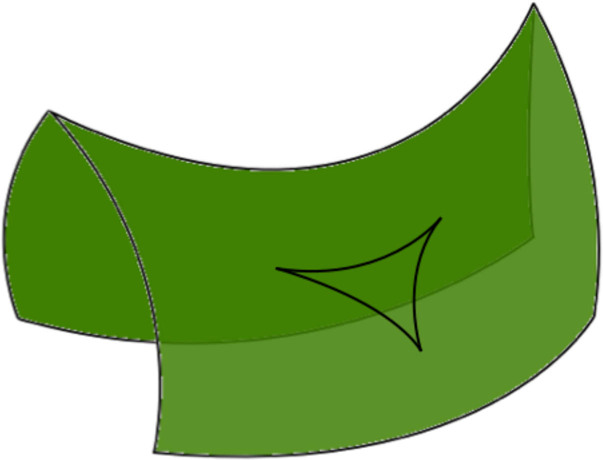
Positive and Negative Curvature

- We have already seen that we can represent the elliptic plane with the surface of a sphere
- Spheres have what mathematicians call **positive** curvature
- It turns out that we can represent the hyperbolic plane with a surface that has **negative** curvature

Positive and Negative Curvature



Positive and Negative Curvature



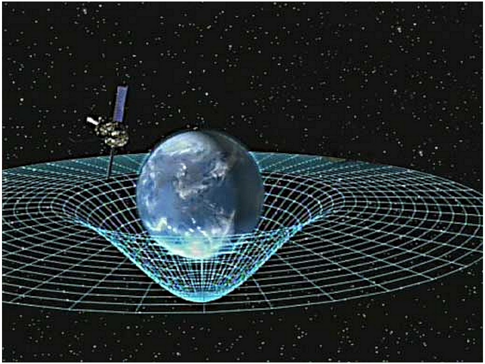
Positive and Negative Curvature

- Euclidean geometry is the geometry of a uniformly flat space
- Hyperbolic geometry is the geometry of a space with uniform negative curvature
- Elliptic geometry is the geometry of a space with a uniform positive curvature
  - **IMPORTANT:** When we describe hyperbolic and elliptic geometries in these ways, we *do not* have to imagine the spaces they are describing as being embedded in a larger, Euclidean space
  - For a helpful introduction, see: Dainton (2010) *Time and Space*, Chapter 13

Variable Curvature

- Euclidean, hyperbolic and elliptic geometry all deal with spaces that have a **uniform** curvature (or lack thereof)
- We can also deal with spaces which are variably curved: perhaps very positively in some places, flat in others, negative in still others
- The mathematics required for all this was invented by Gauss and Riemann in the 19th Century

General Relativity and Variable Curvature



Courtesy of NASA

- It turns out that this mathematics is absolutely crucial to the General Theory of Relativity, which we will come to in Lecture 7

Non-Euclidean Geometry

Which Geometry is Right?

- We have seen that there is a range of different, internally consistent geometries
- Which one is right?
- When it comes to **pure** geometry, this seems like a silly question
  - There is no such thing as the right geometry in this sense, there are just lots of different, equally coherent geometries
- But things seem different when we are dealing with **applied** geometry, i.e. the geometry of the space we actually live in
- It is tempting to say that this is an *empirical* question, to be answered by empirical investigation

Gauss' Thesis that Geometry is Empirical

I am becoming more and more convinced that the [physical] necessity of our [Euclidean] geometry cannot be proved, at least not by human reason nor for human reason. Perhaps in another life we will be able to obtain insight into the nature of space, which is now unattainable. Until then we must place geometry not in the same class with arithmetic, which is purely a priori, but with mechanics.

Gauss, letter to Olbers 1817
Quoted in Kline 1972 p. 872

Gauss' Thesis that Geometry is Empirical

- Gauss is reputed to have tried to test Euclidean geometry by measuring a triangle defined by the peaks of three distant mountains
- This triangle would have been nowhere near large enough to provide a real test
- But we can conceive of other tests, for example we could consider a triangle with Earth at one point, and two distant stars at the other two
  - See Robertson's (1949) for a discussion of Schwarzschild's parallax experiment

Poincaré's Thesis that Geometry is Conventional

- However, this way of thinking about applied geometry was forcefully rejected by Poincaré
- There is no objective matter of fact about the geometry of physical space
- Geometry depends on our conventional choices about how we shall measure
  - What shall we count as a 'straight line'?
  - What shall we count as 'equals'?
- We can choose a geometry on the basis of convenience

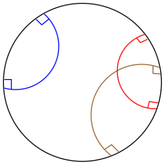
Poincaré's Disk World

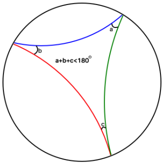
- Imagine that a species of flat people are living on a 2-dimensional, finite disk with radius R
- The disk has a temperature that systematically affects the lengths of all bodies
- The centre of the disk has temperature T, and everywhere else the temperature is calculated as follows:

  $T(R^2 - r^2)$, where r is the distance from the centre
- Measuring rods (and everything else) contract as temperature decreases, and they go all the way to length 0 at the edge where the temperature is 0
- Light is similarly affected, so that it is impossible to tell that things get smaller as we move towards the edge of the disk

Poincaré's Disk World

- The people living in this story would essentially be living in the model of hyperbolic geometry that we discussed earlier





- Because of the way that their measuring instruments shrink, they would think that curved lines are straight

- And when they measure the interior angles of a triangle, their instruments would tell them that those angles add up to less than 180°

Poincaré's Disk World

- Now, in this story, we know that the people living on this disk are wrong
- They think that they live in an infinitely large hyperbolic space, but we know that they really live in a finitely large Euclidean space
- They are just being misled by some weird laws of nature

Poincaré's Disk World

- But we can only say all this because of our position outside of their universe
- From within their universe, it would be impossible to tell whether they were living in a hyperbolic space, or a Euclidean one with weird physical effects
- So, Poincaré concluded, their choice of geometry is a matter of convention, of which convention is simplest for them to work with

Poincaré's Disk World

- The people in the Disk World can choose either of the following conventions
- **Convention 1:** Our measuring rods (etc) remain the same size as we move around
  - Thus we must be living in a non-Euclidean, hyperbolic space
- **Convention 2:** We live in a Euclidean space
  - Thus our measuring rods must be changing shape as we move around

Poincaré's Disk World

- Their choice of convention will have implications for what they take to be straight lines, and whether they think that two lines are the same length
- If the Disk People choose Euclidean space, they will no longer count the 'curved' paths that light takes as straight lines
- Lines that were considered the same length will now be considered different lengths if one of them is closer to the perimeter than the other

What About Us?

- What goes for the Disk People goes for us
- We cannot look at our space from an external God's eye point of view, and no experiment we can perform will tell us what the geometry of our space is
  - It's always an open possibility that some weird law of nature is making our measuring instruments change shape!
- So all we can do is make a choice about which geometry to use
- This is not a completely unconstrained choice: we should choose the geometry that is the simplest, and easiest to use in our universe
- But it is still a conventional choice!

What About Us?

- Or at least, that is what Poincaré thought
- We'll discuss that further in the seminar!

Seminar Reading

- For the seminar, please read:
  - Poincaré, *Space and Geometry* and *Experiment and Geometry*
  - Ben-Menahem, ‘Convention: Poincaré and some of his critics’
- Both are available via the Reading List on the VLE

References

- Kline, M (1972) *Mathematical Thought from Ancient to Modern Times*, vol. 3 (Oxford: OUP)
- Poincaré, H (1905) 'Experiment and Geometry' (Chapter 5 of his *Science and Hypothesis*
- Robertson, H (1949) 'Geometry as a Branch of Physics', pp. 315–32 of Schlipp (1949) *Albert Einstein: Philosopher-Scientist*, 3rd edition

- Helpful Reading:
  - Chapter 13 of Dainton's (2010) *Time and Space* is really useful here
  - And of course, Sklar's (1992) *Philosophy of Physics*, pp. 42–69