Intermediate Logic
Lecture Eight

More Natural Deduction for FOL

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Exercises!

- Present proofs of the following:

    1. $\exists x(Fx \wedge Gx), \forall y(Gy \rightarrow Hy) \vdash \exists x(Fx \wedge Hx)$
    2. $\forall z(Qz \rightarrow Pa) \vdash \exists xQx \rightarrow \exists xPx$
    3. $\exists x\forall yRxy \vdash \forall y\exists xRxy$

More Natural Deduction for FOL

Quantifier Conversion

Identity

Proof-Theoretic Concepts and Semantic Concepts

Derived Rules

- Last week we went through all of the basic rules for the quantifiers
- However, it is helpful to add some **derived** rules, just as we did in TFL
- As before, these derived rules do not add to the power of the system
  - If you can prove something with the help of the derived rules, then you can prove it just using the basic rules
- The derived rules just make it quicker and easier to prove things you could already prove anyway

Conversion of the Quantifiers

$$\begin{array}{l|ll} m & \neg\exists \mathcal{X}\mathcal{A} & \\ & \forall \mathcal{X}\neg\mathcal{A} & \text{CQ, } m \end{array}$$

$$\begin{array}{l|ll} m & \forall \mathcal{X}\neg\mathcal{A} & \\ & \neg\exists \mathcal{X}\mathcal{A} & \text{CQ, } m \end{array}$$

$$\begin{array}{l|ll} m & \neg\forall \mathcal{X}\mathcal{A} & \\ & \exists \mathcal{X}\neg\mathcal{A} & \text{CQ, } m \end{array}$$

$$\begin{array}{l|ll} m & \exists \mathcal{X}\neg\mathcal{A} & \\ & \neg\forall \mathcal{X}\mathcal{A} & \text{CQ, } m \end{array}$$

Deriving the Quantifier Conversion Rules

$\neg\exists x Fx \vdash \forall x \neg Fx$

1	$\neg\exists x Fx$	
2	$\quad Fa$	
3	$\quad \exists x Fx$	$\exists$I, 2
4	$\quad \bot$	$\bot$I, 3, 1
5	$\neg Fa$	$\neg$I, 2–4
6	$\forall x \neg Fx$	$\forall$I, 5

Deriving the Quantifier Conversion Rules

- Now you derive these rules:
  1. $\forall x \neg Fx \ \vdash \ \neg \exists x Fx$
  2. $\exists x \neg Fx \ \vdash \ \neg \forall x Fx$

- And now try this **tough** one:
  3. $\neg \forall x Fx \ \vdash \ \exists x \neg Fx$

An Indirect Strategy

- Universal Introduction is one of the trickiest rules to get your head around
- But now that we have rules for converting quantifiers, we can prove universal generalisations **without** using Universal Introduction!
- This new method isn't always the most elegant way of proving a universal generalisation, but it does always work

An Indirect Strategy

i	$\quad\neg\forall\mathcal{X}\mathcal{A}$	
j	$\quad\exists\mathcal{X}\neg\mathcal{A}$	CQ, i
	$\quad\ldots$	
k	$\quad\bot$	
l	$\neg\neg\forall\mathcal{X}\mathcal{A}$	$\neg$I, i–k
m	$\forall\mathcal{X}\mathcal{A}$	DNE, l

$\forall x(Fx \to Gx) \vdash \forall x(Gx \vee \neg Fx)$

1	$\forall x(Fx \to Gx)$	
2	$\quad Fa$	
3	$\quad Fa \to Ga$	$\forall$E, 1
4	$\quad Ga$	$\to$E, 3, 2
5	$\quad Ga \vee \neg Fa$	$\vee$I, 4
6	$\quad \neg Fa$	
7	$\quad Ga \vee \neg Fa$	$\vee$I, 6
8	$Ga \vee \neg Fa$	TND, 2–5, 6–7
9	$\forall x(Gx \vee \neg Fx)$	$\forall$I, 8

1	$\forall x(Fx \to Gx)$	
2	$\quad \neg\forall x(Gx \lor \neg Fx)$	
3	$\quad \exists x \neg(Gx \lor \neg Fx)$	CQ, 2
4	$\quad\quad \neg(Ga \lor \neg Fa)$	
5	$\quad\quad \neg Ga \land \neg\neg Fa$	DeM, 4
6	$\quad\quad \neg Ga$	$\land$E, 5
7	$\quad\quad \neg\neg Fa$	$\land$E, 5
8	$\quad\quad Fa$	DNE, 7
9	$\quad\quad Fa \to Ga$	$\forall$E, 1
10	$\quad\quad Ga$	$\to$E, 9, 8
11	$\quad\quad \bot$	$\bot$I, 10, 6
12	$\quad \bot$	$\exists$E, 3, 4–11
13	$\neg\neg\forall x(Gx \lor \neg Fx)$	$\neg$I, 2–12
14	$\forall x(Gx \lor \neg Fx)$	DNE, 13

Exercises

- Provide proofs for the following, and feel free to use the new Quantifier Conversion rules:
  1. $\neg\exists x\exists y Lxy \ \vdash \ \neg Laa$
  2. $\forall x(Px \to Qx),\ \forall z((Pz \wedge Qz) \to Rz),\ \neg\exists y\neg Py \ \vdash \ Rb$

More Natural Deduction for FOL

How Would You Prove An Identity?

- Hesperus is the brightest star in the evening sky, and Phosphorus is the brightest star in the morning sky
- It turns out that they are one and the same thing: Hesperus is identical to Phosphorus
- But how would you prove that?
  - You might watch the star move across the sky all night, and see that the same star is the brightest in the evening sky and the brightest in the morning sky
  - You might do some theoretical work, and discover that Venus will inevitably be the brightest object in both the evening sky and the morning sky
  - You might...
- But these are not **logical** ways of proving an identity claim

How Would You Prove An Identity?

- It turns out that there isn't really any purely **logical** way of proving an identity of the form
  - $a = b$
- However, we can always take it as logically given that an object is identical **to itself**
  - $a = a$
  - Hesperus = Hesperus
- This motivates our Introduction Rule for identity

Identity Introduction

$$\begin{array}{|l} c = c \qquad\qquad =\text{I} \end{array}$$

What Can You Infer From An Identity?

- What could you infer from the following identity?
  - Hesperus = Phosphorus
- **Leibniz's Law** tells us that if *a* is identical to *b*, then *a* and *b* must have exactly the same properties
  - If Hesperus is a planet, then Phosphorus must be a planet too
  - If Phosphorus is smaller than Earth, then Hesperus must be smaller than Earth too
- Leibniz's Law quantifies over properties, and we cannot quite do that in FOL
- But we can capture a version of the law in our Elimination Rules for identity

Identity Elimination 1

$$\begin{array}{r|ll} m & \mathcal{a} = \mathcal{b} & \\ n & \mathcal{A}(\ldots\mathcal{a}\ldots\mathcal{a}\ldots) & \\ & \mathcal{A}(\ldots\mathcal{b}\ldots\mathcal{a}\ldots) & \quad =\text{E}, m, n \end{array}$$

- $\mathcal{A}(\ldots\mathcal{a}\ldots\mathcal{a}\ldots)$ is a sentence containing one or more occurrence of the name $\mathcal{a}$
- $\mathcal{A}(\ldots\mathcal{b}\ldots\mathcal{a}\ldots)$ is a sentence obtained by replacing one or more of these occurrences of $\mathcal{a}$ with $\mathcal{b}$

Identity Elimination 2

$$\begin{array}{r|ll} m & a = b & \\ n & \mathcal{A}(\ldots b \ldots b \ldots) & \\ & \mathcal{A}(\ldots a \ldots b \ldots) & \quad {=}\mathrm{E},\ m,\ n \end{array}$$

- $\mathcal{A}(\ldots b \ldots b \ldots)$ is a sentence containing one or more occurrence of the name b
- $\mathcal{A}(\ldots a \ldots b \ldots)$ is a sentence obtained by replacing one or more of these occurrences of b with a

An Example

8	$Fa \vee Rac$	
	...	
15	$a = b$	
16	$Fb \vee Rac$	$=$E, 15, 8

An Example

8	$Fa \vee Rac$	
	...	
15	$a = b$	
16	$Fa \vee Rbc$	$=$E, 15, 8

An Example

8	$Fa \vee Rac$	
	...	
15	$a = b$	
16	$Fb \vee Rbc$	$=$E, 15, 8

An Example

8	$Fb \vee Rbc$	
	...	
15	$a = b$	
16	$Fa \vee Rbc$	$=$E, 15, 8

An Example

8	$Fb \vee Rbc$	
	...	
15	$a = b$	
16	$Fb \vee Rac$	$=$E, 15, 8

An Example

8	$Fb \vee Rbc$	
	$\dots$	
15	$a = b$	
16	$Fa \vee Rac$	$=$E, 15, 8

Proving that Identity is Symmetric

$$\vdash \forall x \forall y (x = y \to y = x)$$

1	$\quad \mid a = b$	
2	$\quad \mid a = a$	$=$I
3	$\quad \mid b = a$	$=$E, 1, 2
4	$a = b \to b = a$	$\to$I, 1—3
5	$\forall y(a = y \to y = a)$	$\forall$I, 4
6	$\forall x \forall y(x = y \to y = x)$	$\forall$I, 5

Proving that Identity is Transitive

$$\vdash \forall x \forall y \forall z((x = y \land y = z) \rightarrow x = z)$$

1	$\quad$ $\underline{a = b \land b = c}$	
2	$\quad$ $a = b$	$\land$E, 1
3	$\quad$ $b = c$	$\land$E, 1
4	$\quad$ $a = c$	$=$E, 2, 3
5	$(a = b \land b = c) \rightarrow a = c$	$\rightarrow$I, 1—4
6	$\forall z((a = b \land b = z) \rightarrow a = z)$	$\forall$I, 5
7	$\forall y \forall z((a = y \land y = z) \rightarrow a = z)$	$\forall$I, 6
8	$\forall x \forall y \forall z((x = y \land y = z) \rightarrow x = z)$	$\forall$I, 7

Exercises

- Provide proofs for the following:
    1. $m = n \vee n = o,\ An \vdash\ Am \vee Ao$
    2. $\forall x\ x = m, Rma \vdash\ \exists x Rxx$
    3. $\forall x \forall y(Rxy \rightarrow x = y) \vdash\ Rab \rightarrow Rba$
    4. $\neg\exists x \neg x = m \vdash\ \forall x \forall y(Px \rightarrow Py)$
    5. $\forall x(x = n \leftrightarrow Mx),\ \forall x(Ox \vee \neg Mx) \vdash\ On$
    6. $\exists x Dx,\ \forall x(Dx \rightarrow x = p) \vdash\ Dp$
    7. $\exists x\big[(Kx \wedge \forall y(Ky \rightarrow x = y)) \wedge Bx\big],\ Kd \vdash\ Bd$
    8. $\vdash Pa \rightarrow \forall x(Px \vee \neg x = a)$

More Natural Deduction for FOL

$\vdash$ versus $\vDash$

- It is **vitally** important not to confuse '$\vdash$' with '$\vDash$'
  - '$\vdash$' expresses provability, and is all about constructing formal proofs according to the rules we have laid out
  - '$\vDash$' expresses validity in FOL, and is all about interpretations and semantics
- Of course, we want there to be some link between '$\vdash$' and '$\vDash$'
- After all, we want to be able to use our formal proofs to test for validity in FOL!

Soundness and Completeness

- **Soundness:**
  - If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash \mathcal{C}$
- **Completeness:**
  - If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vdash \mathcal{C}$

- It turns out that our proof system for FOL is sound and complete
- As a result, we can move back and forth between claims about provability and claims about validity in FOL

The Difference Still Matters!

- But that doesn't mean that the difference between '$\vdash$' and '$\vDash$' isn't important
- '$\vdash$' and '$\vDash$' still **mean** completely different things
- Soundness and completeness results aren't just given, they have to be **proved**
- And in fact, proving the soundness and completeness of a proof system for FOL was **hard**, and it took the genius of Gödel to first do it