The Foundations of Mathematics
Lecture Nine

Structuralism

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Structuralism

Introducing Structuralism

What Numbers Could Not Be

Eliminative Structuralism

Platonic Structuralism

Mathematics as the Science of Structure

- **Structuralism:** Mathematics is the science of *structures*

- Mathematicians don't care about particular systems that instantiate structures
- Mathematicians are interested in the **general structures** that their theories describe

ω-Systems

- Call any well-ordered sequence of things which is exactly as long as the sequence of natural numbers an **ω-system**
- There are lots of different ω-systems, made up of lots of different things
  - **Strokes:** $|, ||, |||, ||||, |||||, \ldots$
  - **Moments:** 12:00:00, 12:01:00, 12:01:30, 12:01:45, 12:01:525, ...
  - **Misc:** $|$, Julius Caesar, $|||, ||||, |||||, \ldots$
- According to structuralism, all of these systems instantiate the number structure, and it is that structure which arithmetic studies

Benefits of Structuralism(?)

- Structuralists claim that their view comes with many advantages
  - **Advantage 1:** It fits well with the actual practice of mathematicians
  - **Advantage 2:** It helps to make the epistemology of mathematics less mysterious
  - **Advantage 3:** It resolves some familiar metaphysical problems about numbers
- But as we will see, there are lots of different versions of structuralism, and they all have their own strengths and weaknesses

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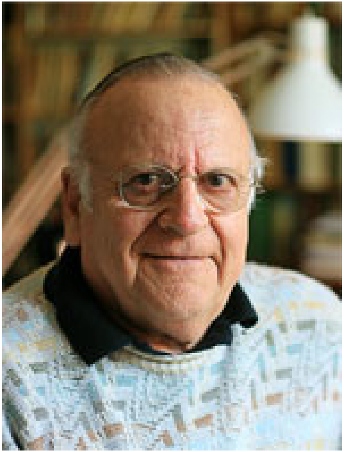
What Numbers Could Not Be

- Benacerraf was an especially influential advocate of structuralism
- He presented an argument for structuralism in his paper 'What Numbers Could Not Be' (reprinted in *B&P*)



Paul Benacerraf

Two Educations

- Imagine we are teaching two children, Johnny and Erica, arithmetic
- Rather than teaching arithmetic in the ordinary way, we decide to teach these children set theory first, and then tell them that the numbers are particular sets

Johnny's Education

- We teach Johnny that the natural numbers are the following sets:

$$0 = \emptyset$$
$$1 = \{\emptyset\}$$
$$2 = \{\{\emptyset\}\}$$
$$3 = \{\{\{\emptyset\}\}\}$$
$$n + 1 = \{n\}$$

(The Zermelo ordinals)

Erica's Education

- We teach Erica that the natural numbers are the following sets:

$$0 = \emptyset$$
$$1 = \{\emptyset\}$$
$$2 = \{\emptyset, \{\emptyset\}\}$$
$$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$
$$n + 1 = n \cup \{n\}$$

(The von Neumann ordinals)

Agreements

- Johnny and Erica can both **count** with their numbers, and they always get the same results
- They can even **prove theorems** about their numbers, and they always get the same results
  - Johnny was told how to understand x *is the successor of* y, $x + y$ and $x \times y$ in terms of his sets, and told to then derive the axioms of PA as theorems of his set theory
  - Erica was told how to understand x *is the successor of* y, $x + y$ and $x \times y$ in terms of her sets, and told to then derive the axioms of PA as theorems of her set theory
  - Of course, the definitions were different, but it all still works

Disagreements

Johnny's Numbers	Erica's Numbers
$0 = \emptyset$	$0 = \emptyset$
$1 = \{\emptyset\}$	$1 = \{\emptyset\}$
$2 = \{\{\emptyset\}\}$	$2 = \{\emptyset, \{\emptyset\}\}$

- **Is 0 a member of 2?**
- **Johnny says: No**
  - $\{\{\emptyset\}\}$ only has one member, which is $\{\emptyset\}$
- **Erica says: Yes**
  - $\emptyset$ is one of the members of $\{\emptyset, \{\emptyset\}\}$
- **Who is right?!**

An Argument for Structuralism

- According to Benacerraf, it is just silly to think that either Johnny or Erica must be right
  - Johnny and Erica's educations were both **good**: they were both taught enough to do arithmetic

- Although Erica and Johnny have both learned about numbers by studying systems of sets, it is just a mistake to think that numbers **are** sets

- Really, the number structure is what is common to the two systems of sets
  - Johnny and Erica's educations were both **bad**: they were both wrongly taught to identify the number structure with particular ω-systems

Indeterminacy

- Recall that an **ω-system** is any well-ordered sequence of things which is as long as the sequence of natural numbers
- According to Benacerraf, the natural number structure is just what's common to all ω-systems, and the 'facts' about the natural numbers are just the facts about what is common to all ω-systems
- Since in some ω-systems it is true that $0 \in 2$, but it is false in others, there is simply no determinate answer to the question of whether $0 \in 2$

Goodbye, Julius Caesar

- Questions about which set the number 2 is should remind you of Frege's **Julius Caesar Problem**

- Recall that Frege was very worried about how we decide the truth-values of sentences like:

  (J) $2 =$ Julius Caesar

- But according to Benacerraf, (J) simply doesn't have a determinate answer

- There are some ω-systems which have Julius Caesar in the 'number 2' position, and some which don't

- So there is no fact of the matter whether 2 is Julius Caesar

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Eliminative Structuralism

- **Eliminative structuralism**: Maths is the science of structures, but there aren't *really* any structures!

  (Eliminative structuralism is sometimes called **in re** structuralism)

- When we "talk about a structure", we are really just generalising over all of the particular systems which have that structure

- **Example:** When we say that there is no largest number in the natural number structure, we are *really* saying: There is no ω-system which has a greatest element

- **General Shape:** $\mathcal{A}(\mathbb{N}) \Rightarrow \forall s(s \text{ is an } \omega\text{-system} \rightarrow \mathcal{A}(s))$

Eliminating Numbers (and their problems)

- It is easy to see why an eliminative structuralist would reject (J) as indeterminate

  (J) $2 =$ Julius Caesar

- Eliminative structuralists don't **really** believe in structures, and so they do not **really** believe in positions in structures

- There is no such thing as the number 2, which would be the 2-position in the number structure

- There are just the various ω-systems which instantiate the number structure, and each of these systems have something plays the role of 2

Epistemological Virtues

- Eliminative structuralism also promises to make it easier to understand how we **know** truths about arithmetic
- These truths are not about some weird, abstract things called numbers, as the platonists claim
- They are about all the systems (including ordinary physical systems) which instantiate the number system
  - There is still a good question about how we know truths about **every** ω-system, but they do not seem as big as the questions that platonists face

A Vacuity Problem

- Imagine that there are only finitely many things, and so there are no ω-systems
- Now consider an obviously false sentence of arithmetic

  (F) $5 + 2 = 8$
- According to eliminative structuralism, (F) means something like this:

  (F′) If s is an ω-system, then if we apply the 'addition' function defined over s to the object in the 5-position of s and the object in the 2-position of s, we will get the object in the 8-position of s
- But if there are no ω-systems, (F′) is vacuously true!
  - (F′) is of the shape: $\forall s(\mathcal{A}(s) \rightarrow \mathcal{B}(s))$
  - Any conditional of that shape is true if nothing is $\mathcal{A}$

One Solution: there are ω-systems!

- One way out of this problem for the eliminative structuralist is to insist that **there are** ω-systems

- But it is not the place of an eliminative structuralist to insist that there are ω-systems made out of physical, concrete objects
  - It is still a real possibility that there are only finitely many physical things
  - Even if there are infinitely many physical things, maths shouldn't be the one to tell us!

- But an eliminative structuralist *could* say that there are ω-systems made out of abstract objects...

Sets to the Rescue?

- Maybe we could posit sets to supply all the systems we need?
- This solution obviously relies on treating sets as proper objects, not as positions in an eliminable structure
  - If we treated set theory in the eliminative structuralists way, then we would face exactly the same problem just described for arithmetic
  - What guarantee do we have that any system instantiates the set structure?
- This is a serious drawback: we now have all the old epistemological problems for the platonist again
  - Sets are abstract objects, so how can we know about them?

Another Solution: go modal!

- Another way out for the eliminative structuralist would be to go modal (see Hellman, *Mathematics without Numbers*)

- Rather than understanding claims about the number structure as generalisations over ω-sequences, we could understand them as generalisations over **possible** ω-sequences

  (F) $5 + 2 = 8$

  (F′′) Necessarily: if s is an ω-system, then if we apply the 'addition' function defined over s to the object in the 5-position of s and the object in the 2-position of s, we will get the object in the 8-position of s

  – **General Shape:** $\mathcal{A}(\mathbb{N}) \Rightarrow \Box\forall s(s \text{ is an } \omega\text{-system} \rightarrow \mathcal{A}(s))$

- So long as it is *possible* for there to be an ω-system, (F′′) will not be vacuously true

Modal Epistemology?

- What is the relevant notion of possibility?
  - Probably not **physical** or **metaphysical**, but **logical**
  - But are we confident that we have a clear grasp on this distinctive kind of **logical** possibility?
- How does an eliminative structuralist know that it is possible for there to be an ω-system?
- This is a serious question because "possible ω-systems" are not the sorts of things we can causally interact with
- So if you were wary about platonism because it claimed that we can know about things we cannot causally interact with, you might be suspicious of this kind of structuralism
  - This should remind you of the problems Field ran into in his **fictionalist** programme

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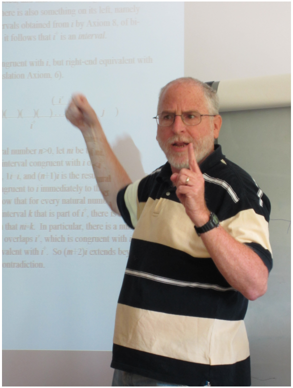
- **Platonic structuralists** take structures to be real objects
- Platonic structuralism is sometimes known as *ante rem* structuralism
- Shapiro is the great advocate of this kind of structuralism



Stewart Shapiro

Putting the Structures back into Structuralism

- Talk about **structures** should not be seen as generalisations over the systems which instantiate those structures
- Talk about structures should be taken at face value, as talk about abstract mathematical objects called 'structures'
  - There is an abstract mathematical object called 'the natural number' structure
- Likewise, talk about **positions** in structures should be taken as talk about abstract objects called 'positions in structures'
  - The number 2 is an abstract mathematical object; it is the 2-position in the natural number structure

Structures as Universals

- Platonic structuralists think about **structures** in much the same way as platonists think about **universals**
- Universals are meant to be the grounds of objective resemblance
  - Why do all red things resemble each other? Because they all instantiate the universal **red**
- Platonists think of universals as abstract objects which "transcend" the objects that instantiate them
- For a platonic structuralist, structures are structural universals, instantiated by systems of objects

Positions and Position-Holders

- The platonic way of thinking about structures may at first sound odd, but it isn't so far removed from our ordinary ways of thinking
- We sometimes use the term 'the goalkeeper' to refer to the particular person who is playing the position of the goalkeeper
  - The goalkeeper is terrible today!
- But we **also** use 'the goalkeeper' to refer to the position of goalkeeper itself
  - The goalkeeper is the only player that can pick the ball up
- Here we seem to be treating the position of goalkeeper as its own, abstract object, over and above any particular thing which is in that position

The Number 2 as an Office

- According to platonic structuralism, the number 2 itself is a **position** in a structure
  - If we say something like '$\{\{\emptyset\}\}$ is 2 in the Zermelo ordinals', we are saying that $\{\{\emptyset\}\}$ **holds** the position of 2 in the Zermelo ordinals
- So when we say something like '2 is prime', we are describing something about the **position** of 2
  - In particular, we are saying that this position is related to the other number positions in such a way that it is divisible only by the 2-position and the 1-position

Problem Solved!

- **Platonic structuralism** neatly avoids the problem for **eliminative structuralism**
- According to platonic structuralism, talk about a structure isn't a generalisation over systems instantiating that structure, and so wouldn't be vacuous if there were no such systems
- And in fact, there is *always* a guarantee that every structure is instantiated by some system
  - The natural number structure itself is an ω-system
  - It is an infinite progression of objects: the positions in the number structure
  - So the natural number structure instantiates itself!

The Drawbacks of Platonic Structuralism

- The drawback to platonic structuralism is obvious: it is a **platonic** theory, and so faces all the problems of platonism
- **Epistemological**
  - Structures and the positions in structures are **abstract** objects, so how can we know about them?
- **Semantic**
  - Structures and the positions in structures are **abstract** objects, so how can we refer to them?
- **Metaphysical**
  - Structures and the positions in structures are abstract **objects**, so why can we dismiss the Julius Caesar Problem?

Epistemology: Abstraction

- Shapiro offers a sophisticated account of the **epistemology** of structures
- We start with particular systems instantiating simple structures
  - e.g. two apples side by side which instantiate the 2-structure (the structure of numbers up to 2)
- By reflecting on these simple systems, we somehow **abstract** the structures they are instantiating
- However, abstraction will only take us to very small structures
  - It won't even take us to the 26,382-structure

Epistemology: Projection

- Shapiro also thinks that we can **project** from simpler structures to more complex structures
- We notice that, when n is low, an n-structure can be extended into an $(n+1)$-structure
  - We can go from a 3-structure to a 4-structure
- We then take a step of projection to infer that, for each n, there is an $(n+1)$-structure
- We then take another step of projection, and infer that there is an ω-structure

Epistemology: Implicit Definition

- Shapiro is clear that abstraction and projection won't get us very far, and so he leans heavily on **implicit definition**
- The idea is that every coherent mathematical theory describes a structure
- **Question:** What exactly does coherent mean?
  - For Shaprio, 'coherent' is a primitive undefinable concept
  - However, he also believes that a second-order theory is coherent iff it can be modelled in set theory
- Thus we can come to know about mathematical structures by considering coherent mathematical theories

Does this Epistemology Work?

- MacBride challenges Shapiro's epistemology in 'Can "ante rem" structuralism solve the Access Problem?'
  - MacBride grants for the sake of argument that we can abstract simple platonic structures from simple systems

    (Given the criticisms MacBride goes on to make, I'm not entirely clear why he is willing to grant that much)
  - However, he thinks that Shapiro's reliance on projection and implicit definition reveals that he doesn't have a solution to the Access Problem
- Shapiro replies in 'Epistemology of mathematics: What are the questions? What count as answers?'
  - Shapiro claims that MacBride's objections presuppose some sort of foundationalist epistemology
  - However, Shapiro is something like a Quinean holist

Semantics

- If you like Shapiro's **epistemology**, you could give a similar story about the **semantics** of platonic structures
- We initially come to refer to simple structures by introducing them as abstractions from simple systems
- We then introduce talk of more complex structures via projection
- We then introduce talk about *even more* complex structures via implicit definition

Metaphysics

- According to the platonic structuralist **metaphysics**, numbers are genuine objects
- Why, then, can platonic structuralists just dismiss the Julius Caesar Problem? (Is $2 =$ Julius Caesar?)
- **Shapiro's answer:** Numbers are positions in structures, and the properties of a position are determined by its place in the structure
- But while we apprehend a position in a structure by appreciating the way it sits in the structure, does that mean we can automatically conclude that there is no more to a position than that?
  - Even Shapiro would like to identify 2 in the natural number structure with 2 in the real number structure

For the Seminar

- For the final seminar, please read:
  - Benacerraf, ‘What numbers could not be’
  - Shapiro, *Philosophy of Mathematics: Structure and Ontology*, chs 3 & 4

- You can find these chapters on the VLE

With Thanks to Mary Leng

I would like to thank Mary Leng, who let me borrow freely from her notes in preparation for these lectures