The Foundations of Mathematics
Lecture Eight

Field's Programme

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Field's Programme

Benacerraf's Dilemma

- **Semantics** pushes us towards **platonism**
  - A good semantics for mathematics will treat existential claims in maths in the same way that they are treated in other parts of our language
  - So since 'There is a prime number between 4 and 6' is true, we are committed to the existence of prime numbers

- **Epistemology** pushes us towards **nominalism**
  - A good epistemology for mathematics will tell us how we know so many mathematical truths
  - But our mathematical knowledge would be utterly mysterious if mathematical truths were about abstract objects outside of spacetime

The Indispensability Argument

(1) We ought to believe in all of the entities that are indispensable to our best scientific theories

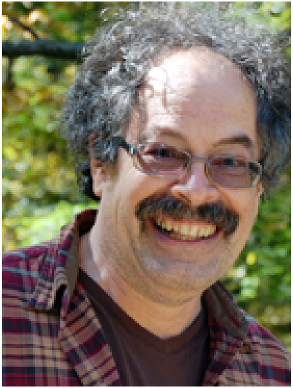
(2) Mathematical entities are indispensable to our best scientific theories

∴ (3) We ought to believe in mathematical entities

Field's Programme

Field's Fictionalism

- Field is a nominalist: he does not believe in mathematical objects
- But Field also thinks that our mathematical theories are ontologically committed to mathematical objects
  - Our mathematical theories can be true only if mathematical objects exist
- So Field concludes that our mathematical theories are false!
- Field's position is known as **fictionalism**



Hartry Field

Field on Benacerraf's Dilemma

- Field entirely agrees that our **epistemology** pulls us towards **nominalism**
  - Field is a nominalist **because** he cannot see how we could know about (or refer to) abstract objects
- But he denies that our **semantics** pulls us towards **platonism**
  - Field agrees that we should read 'There is a prime number between 4 and 6' as a *bona fide* existential claim
  - So **if** 'There is a prime number between 4 and 6' is true, **then** numbers exist
  - But Field denies that it is true!

Field against the Indispensability Argument

- Field agrees that we should believe in all of the entities that are indispensable to our best scientific theories

- But Field denies that mathematical entities are indispensable to our best scientific theories

- Although scientists standardly use mathematics in their theories, you could formulate **nominalistic** versions of those theories which do not mention any numbers at all

Why Should We Trust Mathematics?

- Although Field thinks that we *could* present nominalistic versions of our scientific theories, he is happy to admit that the mathematical theories are much easier to work with
- This raises a **big** question for fictionalism:
  - If our mathematical theories are false, then why should we trust mathematical science?
- **Field's answer:** because mathematics is **conservative** over our nominalistic theories
  - M is conservative over N iff: for any nominalistic sentence $\mathcal{A}$, M+N entails $\mathcal{A}$ iff N entails $\mathcal{A}$

What is the Point of Mathematics?

- According to Field, mathematics is a **useful** fiction
- Mathematised science is much easier to work with than nominalistic science
  - Using mathematics makes it a lot easier to figure out the consequences of our nominalistic claims
- But because our mathematical theories are conservative over nominalistic theories, we know that mathematised science will never mislead us

A Theoretical Juice Extractor

in the establishment of empirical knowledge, mathematics (as well as logic) has, so to speak, the function of a theoretical juice extractor: the techniques of mathematical and logical theory can produce no more juice of factual information than is contained in the assumptions to which they are applied; but they may produce a great deal more juice of this kind than might have been anticipated upon a first intuitive inspection of those assumptions which form the raw material for the extractor.

(Carl Hempel, 1945, 'On the nature of mathematical truth', p.391)

Field's Programme

Science without Numbers

- In his landmark book, *Science without Numbers*, Field starts to work through some of the details of his **fictionalist programme**

- He tried to do three things:
  (1) Provide a nominalistic version of Netwonian Gravitational Theory
  (2) Provide bridge laws between this nominalistic theory and mathematics
  (3) Prove that mathematics is conservative over this nominalistic theory

Mathematical Newtonian Gravitational Theory

- Ordinary Newtonian Theory makes free reference to **mathematical entities**
- We represent individual spacetime co-ordinates with **quadruples of real numbers**, $\langle x, y, z, t \rangle$
- We represent regions of spacetime with **sets of quadruples of real numbers**, $\{\langle x, y, z, t \rangle : \ldots\}$
- We then use **mathematical relations and functions** to express the gravitational theory
  - For example, 'the gravitational potential at $\langle x, y, z, t \rangle$' expresses a function from quadruples of real numbers to real numbers

Spacetime Points and Regions

- Field's nominalistic Newtonian theory gets rid of all these **mathematical entities**
- Rather than using quadruples of real numbers to represent spacetime points, the theory just quantifies over the **spacetime points** themselves (which it treats as real objects)
- Rather than using sets of quadruples of real numbers to represent spacetime regions, the theory just quantifies over the **spacetime regions** themselves (which it treats as real objects)

Physical Relations

- Rather than expressing the theory in terms of mathematical relations, it uses **physical relations** that hold between spacetime points and regions
- **Example**
  - Field does not use 'the gravitational potential of x', which expresses a function to real numbers
  - Instead, he uses 'the difference in gravitational potential between x and y is less than that between z and w', which expresses a *physical* relation between spacetime points

Nominalistic Newtonian Gravitational Theory

- Field's nominalistic Newtonian Gravitational Theory is **empirically equivalent** to ordinary, mathematical Newtonian Theory
- So on the face of it, Field has completed his first task:
  (1) Provide a nominalistic version of Netwonian Gravitational Theory
- Now we turn to his second task:
  (2) Provide bridge laws between this nominalistic theory and mathematics

Representation Theorems

- Field explains how mathematics can be applied to his Nominalistic Newtonian Theory by proving some **representation theorems**
  - We prove that there's a function, f, from spacetime points to quadruples of real numbers, s.t. relations between spacetime points can be represented by relations between these quadruples
  - **Example:** There is a function, g, s.t. $|g(f(x)) - g(f(y))| < |g(f(z)) - g(f(w))|$ iff the difference in gravitational potential between x and y is less than the difference in gravitational potential between z and w
- We obviously have to use **mathematics** to prove these theorems, but that is not a problem if mathematics really is conservative over the nominalised Newtonian theory

Conservativeness: A Philosophical Argument

- We come now to Field's final task:
  (3) Prove that mathematics is conservative over Nominalistic Newtonian Theory
- Field offers a **philosophical** reason for accepting (3):
  - Even a **platonist** should admit that pure mathematics is conservative over every nominalistic theory
  - Otherwise, pure mathematics would have to imply some (non-trivial) nominalistic claim
  - But how could a truth about **abstract objects** possibly imply anything about **concrete physical objects**?!

Conservativeness: A Set-Theoretic Proof

- Field also offers a **set-theoretic proof** of conservativeness
  - Given appropriate background assumptions, you can model a consistent nominalistic theory in a proper subdomain of set theory, and model the set theory using the rest of the domain
  - So set theory must be consistent with every consistent nominalistic theory, which implies that it must be conservative over every nominalistic theory

    (Suppose M+N entail nominalistic sentence $\mathcal{A}$, but N doesn't alone; in that case, $N \cup \{\neg\mathcal{A}\}$ is a consistent nominalistic theory inconsistent with M)

- Isn't it odd for a **nominalist** like Field to offer this set theoretic proof?

- Field's official line is that his argument is meant to convince **platonists** that there is nothing wrong with nominalism

Field's Programme

How are Spacetime Points better than Numbers?

- There is a very clear sense in which Field simply replaces quadruples of real numbers with **spacetime points**
- How much of an improvement that is, from a **nominalist** point of view?
- On the face of it, spacetime points are not physical objects, and they are not things that we an interact with
- So how exactly are spacetime points any better than numbers?
- Indeed, some philosophers (**relationists**) have denied that spacetime points exist, for broadly the same reasons that philosophers have denied that numbers exist

Interacting with Spacetime Points

- Field's response to this objection is to insist that, according to modern physics, we **do** causally interact with spacetime points

- Physics posits fields (e.g. electromagnetic fields) which are spread out throughout spacetime

- According to Field, when a physicist says that an electromagnetic field has a given strength at a given spacetime point, we should understand them as attributing an electromagnetic strength **to the spacetime point**

  (And if you really want to distinguish fields from spacetime, Field can just swap out spacetime points for points in the fields...)

- So spacetime points have causal powers, which is how we can interact with them, and thus know about them

Other Scientific Theories

- Let's grant that Field has successfully nominalised Newtonian Gravitational Theory
- That is just one (out of date!) scientific theory, and Field must nominalise **every** scientific theory we accept
- Moreover, it is not at all clear that Field will be able to use his nominalisation of Newtonian Theory as a template
  - In the Newtonian case, it was a simple matter of replacing quadruples of real numbers with spacetime points
  - But other theories (e.g. quantum mechanics) use very sophisticated, abstract mathematics

Field's Programme

Field versus Hilbert

Hartry Field

- There are clear similarities between Field's fictionalist programme and Hilbert's formalist programme
- Hilbert's programme was scuppered by Gödel's Incompleteness Theorems
- The Incompleteness Theorems pose a threat to Field's programme too

David Hilbert

Field and Hilbert: The Similarities

- **Hilbert's Programme**
  - Finitary Mathematics is a body of truths
  - Ideal Mathematics is a meaningless game
  - But Ideal Mathematics is still useful, but it is conservative over Finitary Mathematics
- **Field's Programme**
  - Nominalistic science is a body of truths
  - Mathematics is ontologically committed to entities that do not exist, and so is false
  - But mathematics is still useful, but it is conservative over nominalistic science

Gödel's Incompleteness Theorems

- Hilbert's theory was destroyed by Gödel's Theorems
- Shaprio showed that those theorems also cause trouble for Field in his 'Conservativeness and Incompleteness'

Modelling Robinson Arithmetic

- Consider a line of spacetime points in Field's nominalistic Newtonian Theory
- This line has all of the structure of the mathematical real number line
- Consequently, we can model the natural number line within that line
  - The natural number line just has less structure than the real number line
- Moreover, we can formulate versions of all of the axioms of Robinson Arithmetic which apply to this model of the natural number line

Incompleteness and Conservativeness

- If we assume that Field's nominalistic theory (N) is recursively axiomatisable, then we can code the claim that N is consistent into a nominalistic claim about spacetime points, Con_{N}
- **Gödel's Second Incompleteness Theorem** then implies that if N is consistent, then N cannot prove its own consistency
  - $\mathrm{N} \nvdash Con_{\mathrm{N}}$
- However, if M includes set theory, we can use M to prove that N is consistent:
  - $\mathrm{M}+\mathrm{N} \vdash Con_{\mathrm{N}}$
- So, since Con_{N} is a nominalistic claim, M is not conservative over N!

Two Notions of Entailment

- Earlier I told you that Field **proved** that mathematics is conservative over N
- But conservativeness is defined in terms of **entailment**:
  - M is conservative over N iff: for any nominalistic sentence $\mathcal{A}$, M+N entails $\mathcal{A}$ iff N entails $\mathcal{A}$
- As I explained in Lecture 5, there are two notions of entailment:
  - **Syntactic deducibility:** $\Gamma \vdash \mathcal{A}$ iff there is a proof of $\mathcal{A}$ from premises in Γ
  - **Semantic logical consequence:** $\Gamma \vDash \mathcal{A}$ iff no interpretation makes all of the sentences in Γ true and $\mathcal{A}$ false

Two Notions of Conservativeness

- So there are two different versions of conservativeness:
  - M is **deductively conservative** over N iff: for any nominalistic sentence $\mathcal{A}$, M+N $\vdash \mathcal{A}$ iff N $\vdash \mathcal{A}$
  - M is **semantically conservative** over N iff: for any nominalistic sentence $\mathcal{A}$, M+N $\vDash \mathcal{A}$ iff N $\vDash \mathcal{A}$
- What Field proved was that mathematics is **semantically conservative** over N
- What Shapiro proved (using Gödel's Theorems) was that mathematics is not **deductively conservative** over N

But How Is That Possible?

- You might be wondering how mathematics could be semantically conservative but not syntactically conservative
- (Some versions of) Field's Nominalistic Newtonian Theory was **second-order**
- And as we saw in Lecture 5, the standard deductive system of second-order logic is not complete relative to the standard semantics:
  - It is not the case that for every sentence $\mathcal{A}$ and every set of sentences Γ: if $\Gamma \vDash \mathcal{A}$ then $\Gamma \vdash \mathcal{A}$
- In other words, a second-order theory can be syntactically consistent, and yet lack a model

How should we Interpret the Second-Order Quantifiers?

- Second-order quantifiers are often interpreted as quantifying over sets, but that obviously isn't an option for a nominalist
  - $\exists X\ Xa \Rightarrow$ there is some set X such that $a \in X$
- Field suggests we interpret them as quantifying over regions of spacetime points (thought of as mereological sums)
  - $\exists X\ Xa \Rightarrow$ there is some region X such that a is in (i.e. is a part of) X
- Mereological sums of concrete objects are themselves concrete objects
- So if Field is right that spacetime points are concrete objects that we can interact with, then we can interact with spacetime regions

Field and Second-Order Logic

- However, Field is still a bit wary about using second-order logic
- This is a theorem of second-order logic:
  - $\forall x \forall y \exists Z (Zx \wedge Zy)$
- For Field, this says that for any two spacetime points, x and y, there is always a region which contains them both
- So it looks like pure logic is **introducing** commitment to regions, and Field thought logic shouldn't introduce *any* ontological commitments

Field and Logical Consequence

- It also isn't clear whether Field can really help himself to the notion of **semantic** logical consequence
- Logicians usually define this notion in terms of **sets**, but Field doesn't really believe in mathematical entities
- Field tries to get around this problem by defining logical consequence in terms of **logical possibility**
  - $\mathcal{A} \vDash \mathcal{B}$ iff $\neg\Diamond(\mathcal{A} \wedge \neg\mathcal{B})$
- Field then suggests that we should just take this notion of logical possibility as primitive

Field and Logical Consequence

- However, it really is not clear whether a **nominalist** is allowed to take logical possibility as primitive
- We have no way of causally interacting with the merely "logically possible"
- So how can we know anything about logical possibility?
  - Remember, Field is distinguishing logical possibility from **syntactic consistency**
- In particular, consider the claim: $\Diamond$(there are infinitely many objects)
- How could Field ever know anything like that?

Return to First-Order?

- You might suggest that Field return to a first-order setting, since first-order logic is sound and complete
  - In first-order logic: $\Gamma \vdash \mathcal{A}$ iff $\Gamma \vDash \mathcal{A}$
- In a first-order setting, Field's argument for semantic conservativeness therefore also establishes syntactic conservativeness
- So, since $\mathsf{N} \nvdash Con_{\mathsf{N}}$, $\mathsf{M}+\mathsf{N} \nvdash Con_{\mathsf{N}}$

Goodbye Representation Theorems

- However, Shapiro points out that in a first-order setting, Field loses out on his **representation theorems**
- Con_{N} is a nominalistic sentence which codes the claim that N is consistent
- But we can also construct a *mathematical* sentence that codes N's consistency, Con^{*}_{N}
- We can also demonstrate that (if M includes set theory), that $\mathrm{M}+\mathrm{N} \vdash Con^{*}_{\mathrm{N}}$
- It follows that $\mathrm{M}+\mathrm{N} \nvdash Con_{\mathrm{N}} \leftrightarrow Con^{*}_{\mathrm{N}}$, which is possible only if we can't establish that spacetime points can be represented by quadruples of real numbers

For the Seminar

- Required reading:
  - Field, 'Realism and Anti-Realism about Mathematics', ch. 2 in his *Realism, Mathematics and Modality*
  - Balaguer, *Platonism and Anti-Platonism in Mathematics*, ch.7
- You can find links to both of these on the VLE

For Next Week

- Next week we will be looking at structuralism
- Please read the following
  - *Shapiro* ch. 10
  - Benacerraf, 'What Numbers Could not Be' in *B&P*