The Foundations of Mathematics
Lecture Four

Hilbert's Programme

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A Reminder of why Mathematics is Special

- In Lecture 1, we pointed out that mathematical truths seems to have some special properties:
  - Mathematical truths are **necessarily** true
  - Mathematical truths can be known **a priori**
  - Mathematical truths can be known with **certainty**
- Add on top of all of this that mathematics seems to deal with **infinities**
  - There are infinitely many natural numbers, more real numbers, *even more* sets of real numbers. . .
- Any good philosophy of maths needs to say something about these special features of mathematics

Formalism: What?

- According to **formalism**, mathematics isn't really about "numbers" or "sets"
- Mathematics is just about manipulating **symbols**
- So if there is any such things as mathematical "truths" (a big *if!*), then they are just truths about how we can manipulate symbols

Formalism: Why?

- Formalism is attractive for (at least) two reasons
- **First:** it seems to fit very well with the actual practice of mathematics
  - Mathematicians seem to spend most of their time manipulating formulas and equations
- **Second:** it promises to deflate the big philosophical questions about mathematics
  - There is no mysterious necessary, a priori certain truth about numbers and the like; there are just simple truths about which symbols can be derived from which symbols

Term-Formalism

- **Term-formalism:** mathematics is about **symbols**
- According to term-formalism, arithmetic is about **numerals**, like '2' and '3'
  - Platonists think of numerals as symbols which refer to numbers
  - Term-formalists think of numerals as symbols which refer to themselves
  - Term-formalists identify numbers with numerals
- For term-formalists, numbers are not metaphysically mysterious; they are just symbols we manipulate

Two Problems for Term-Formalism

- **Problem 1:** What does '$1 + 2 = 3$' mean?
  - Distinguish between **types** and **tokens**: '1' and '1' are two *tokens* of the same symbol *type*
  - '$1 = 1$' can just mean ' "1" is the same type of symbol as "1" '
  - But '$1 + 2 = 3$' cannot mean ' "$1 + 2$" is the same type of symbol as "3" '

- **Problem 2:** What about non-denumerable infinities?
  - A set is **denumerable** iff it is no bigger than the set of natural numbers, $\{0, 1, 2, 3 \dots\}$
  - It is generally assumed that languages only contain a denumerable infinity of (types of) symbol
  - So how can term-formalism accommodate non-denumerable infinities, like the set of real numbers?

Game-Formalism

- **Game-formalism:** mathematics is not *about* anything

- According to game-formalism, mathematics is a meaningless **game** that we play with symbols

- Game-formalism is very different from term-formalism
  - According to term formalism, there *are* such things as mathematical truths; they are just truths about symbols
  - According to game-formalism, there *are no* mathematical truths; mathematics isn't in the business of expressing propositions, true or false

- For game-formalists, numbers are not metaphysically mysterious; they aren't anything at all!

A Problem for Game-Formalism

- The big challenge for game-formalism is to explain why the mathematical game that we play can be **applied** so usefully to the real world
- If mathematics is just a game, then it is non-compulsory: we **could** have played a different game
- For example, rather than playing our game of arithmetic, we could have played a variant in which $5 + 2 = 8$
- Why is it that we can use the arithmetic game we actually play to build bridges which stand up, but not this variant game?

A Problem for Game-Formalism

an arithmetic without thought as its content will also be without possibility of application. Why can no application be made of a configuration of chess pieces? Obviously, because it expresses no thought. If it did so and every chess move conforming to the rules corresponded to a transition from one thought to another, applications of chess would also be conceivable.

A Problem for Game-Formalism

Why can arithmetical equations be applied? Only because they express thoughts. How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules? It is applicability alone which elevates arithmetic from a game to the rank of a science

(Frege 1903, Grundgesetze vol.2, §91)

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David Hilbert

- Hilbert was one of the greatest mathematicians of the 19th and 20th Centuries
- He was also one of the most influential formalists
- He tried to combine term-formalism with game-formalism to make something stronger

David Hilbert

Hilbert's Suspicion of the Infinite

- Hilbert was deeply suspicious of the infinite, and that suspicion had two roots
- **The Logical Paradoxes**
  - Early developments of infinitary set theory were beset by paradoxes (e.g. Russell's Paradox)
  - This raised the question: How can we be confident about this new branch of mathematics?
- **No Completed Infinities in Nature**
  - It is not obvious that the Universe is infinitely big, or infinitely divisible
  - Even if it is, this infinity appears to be merely **potential**, not **actual**

Hilbert's Suspicion of the Infinitely Small

a homogenous continuum which admits of the sort of divisibility needed to realize the infinitely small is nowhere to be found in reality. The infinite divisibility of a continuum is an operation which exists only in thought. It is merely an idea which is in fact impugned by the results of our observations of nature and of our physical and chemical experiments.

(Hilbert, 'On the infinite', in B&P, p.186)

Hilbert's Suspicion of the Infinitely Big

The attempt to prove the infinity of space by pure speculation contains gross errors. From the fact that outside a certain portion of space there is always more space, it follows only that space is unbounded, not that it is infinite. Unboundedness and finiteness are compatible. In so-called elliptical geometry, mathematical investigation furnishes the natural model of a finite universe.

(Hilbert, 'On the infinite', in B&P, p.186)

Hilbert's Suspicion of the Infinite

> *the infinite is nowhere to be found in reality. It neither exists in nature nor provides a legitimate basis for rational thought.*
>
> *(Hilbert, 'On the infinite', in B&P, p.201)*

Hilbert's Love of Modern Mathematics

- Despite his suspicion of the infinite, Hilbert thought that modern, infinitary mathematics was one of humanity's greatest intellectual achievements

  "mathematical analysis is a symphony of the infinite" (Hilbert 'On the infinite', p.187)

  "No one shall drive us out of the paradise which Cantor has created for us." (ibid, p.191)

- How can Hilbert reconcile these two opposed attitudes?

Finitary Mathematics vs. Ideal Mathematics

- Hilbert divided mathematics into two broad categories:
- **Finitary Mathematics**
  - The core of mathematics which is in no way committed to the existence of a completed infinity (i.e. infinitely big collections)
- **Ideal Mathematics**
  - All the rest of mathematics
- Hilbert was a term-formalist about Finitary Mathematics, and a game-formalist about Ideal Mathematics

Half-Term, Half-Game Formalism

- The idea is that Finitary Mathematics is a body of **truths** about the symbols we use in mathematics, their properties and the relations they bear to each other
- Ideal Mathematics is then a **meaningless game** we play on top of the Finitary Mathematics
- But although Ideal Mathematics is meaningless, it is not **pointless**
  - Ideal Mathematics is useful because it helps us to derive results in Finitary mathematics

The Best of Both Worlds?

- **Non-denumerable infinities**
  - Pure term-formalists struggle with branches of mathematics that concern non-denumerable infinities
  - But not Hilbert! He simply says that those branches of mathematics are branches of **Ideal Mathematics**
- **Applications**
  - Pure game-formalists struggle to explain why mathematics is applicable
  - But not Hilbert! Ideal Mathematics may just be a game, but the point of the game is to help us derive results in meaningful **Finitary Mathematics**

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What are Hilbert's Terms?

- According to Hilbert, Finitary Mathematics is a body of truths about mathematical symbols
- He suggested that instead of Arabic numerals like '2' and '3', we use finite strings of strokes as our numerals:
  - $1 =_{df} |$
  - $2 =_{df} ||$
  - $3 =_{df} |||$

  (Hilbert's numerals were a bit like Roman numerals, but he never used numerals like 'V' or 'C': it's just longer and longer strings of '|')
- Hilbert's numerals were **types**, not **tokens**

The Basic Finitary Truths

- The basic finitary truths are particular equalities and inequalities, like the following:
  - $|| = ||$
  - $||| > ||$
  - $|| + | = |||$
- Here is how you are meant to understand them:
  - The symbol '$||$' is the same as the symbol '$||$'
  - The symbol '$||$' is a proper part of the symbol '$|||$'
  - If you write the symbol '$||$' followed by the symbol '$|$', then you get the symbol '$|||$'

Finitary Intuition

- How do we know these basic finitary truths?
- They are meant to be something we can directly intuit
  - We can just *see* that '||' is a proper part of '|||'
- Kantian intuition is intuition of space and time, but Hilbert thinks that the structure of space and time is *a posteriori*
- It seems plausible to suggest that Hilbert believes in a distinct kind of **finitary intuition**

Finitary Intuition

The objects of number theory are for me — in direct contrast to Dedekind and Frege — the signs themselves, whose shape can be generally and certainly recognized by us — independently of space and time, of the special conditions of the production of the sign, and of insignificant differences in the finished product. The solid philosophical attitude that I think is required for the grounding of pure mathematics — as well as for all scientific thought, understanding, and communication — is this: In the beginning was the sign.

(Hilbert, in From Brouwer to Hilbert, p.202)

Bounded Generalisations

- Finitary Mathematics also includes **bounded generalisations**, i.e. generalisations over numbers below a given finite bound
  (1) There is a twin prime smaller than 5
  (2) Every prime number smaller than 10 is a divisor of 210

- You can check whether bounded generalisations are true in a finite period of time

- You can write them as finite conjunctions/disjunctions
  (1*) Either 0 is a twin prime, or 1 is a twin prime, or . . ., or 5 is a twin prime
  (2*) 2 is a divisor of 210, and 3 is a divisor of 210, and 5 is a divisor of 210, and 7 is a divisor of 210

No Unbounded Existential Generalisations

- Finitary Mathematics **does not** include any *unbounded* existential generalisations

  (3) There is a greatest twin prime

- You cannot write this as a finite disjunction

  (3*) Either 0 is the greatest twin prime, or 1 is the greatest twin prime, or ...

- There is no guarantee that you could check whether (3) is true in a finite period of time

Unbounded Universal Generalisations as Classically Interpreted

- Unbounded universal generalisations are classically interpreted as generalisations over **infinitely many** numbers

(4) Every number is odd or even

(4*) 0 is odd or even, and 1 is odd or even, and ...

- When we understand unbounded universal generalisations in this way, they obviously have no place in **Finitary Mathematics**

Unbounded Universal Generalisations as Schemes

- However, we can include unbounded universal generalisations in **Finitary Mathematics** if we interpret them as **schemes**

  (5) $\mathfrak{a} + 1 = 1 + \mathfrak{a}$

- Schemes are *forms* of sentence; an **instance** of a scheme is a sentence of that form
  - To make an instance of (5), just replace '$\mathfrak{a}$' with a numeral

- When you "assert" (5), you are really taking on the commitment to accept any of its instances when you are presented with them

- You may "assert" a scheme when you have a finitary procedure for proving any given instance of the scheme

Negating Universal Generalisations

- Finitary schemes cannot be **negated**
- The negation of an ordinary universal generalisation is equivalent to an **unbounded** (i.e. infinitary) existential generalisation

  (6) $\forall n{>}2\,(En \to \exists j{<}n\,\exists k{<}n\,(Pj \wedge Pk \wedge n = j + k))$

  (¬6) $\exists n{>}2\,(En \wedge \neg\exists j{<}n\,\exists k{<}n\,(Pj \wedge Pk \wedge n = j + k))$

- Remember, asserting a scheme just signals a willingness to assert any instance you are presented with
- That is not something you could negate!
  - You can have a scheme like this: $\neg(\mathfrak{a} + 1 = 1 + \mathfrak{a})$
  - But this is another universal scheme: $\mathfrak{a} + 1$ is *always* distinct from $1 + \mathfrak{a}$

Finitary Mathematics and Intuitionism: Similarities

- We can now see that Classical Logic **does not hold** within finitary mathematics
- If $\mathcal{A}$ is a finitary universal generalisation, then $\mathcal{A} \vee \neg\mathcal{A}$ will not be true
  - $\neg\mathcal{A}$ will be the negation of a finitary universal generalisation, and so will not be a meaningful finitary claim
  - As a result, the whole disjunction $\mathcal{A} \vee \neg\mathcal{A}$ will not be a meaningful finitary claim
- Similar rules will have to be restricted, like Double Negation Elimination and Reductio Ad Absurdum
- This is obviously very reminiscent of **intuitionism**, but we mustn't take the analogy too far

Finitary Mathematics and Intuitionism: Differences

- **Difference 1: Metaphysics**
  - Intuitionists think of natural numbers as **mental constructions**
  - Hilbert think of natural numbers as types of symbol
- **Difference 2: Meaning**
  - Intuitionists refuse to assert some instances of $\mathcal{A} \vee \neg\mathcal{A}$, but do not deny that they are all *meaningful*
  - Hilbert says that some instances of $\mathcal{A} \vee \neg\mathcal{A}$ are **meaningless**
- **Difference 3: Logic**
  - Intuitionists reject Classical Logic
  - Hilbert restricts Classical Logic within Finitary Mathematics, but retains it for Ideal Mathematics

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Introducing Primitive Recursive Arithmetic

- There is disagreement about what *exactly* counts as Finitary Mathematics
- But lots of people identify Finitary Mathematics with **Primitive Recursive Arithmetic** (PRA)
- PRA is a weak theory of arithmetic
  - It is strictly weaker than Peano Arithmetic, which we'll discuss next week

PRA: The Background Logic

- The background logic of PRA is between TFL and FOL
  - The logic of PRA includes predicates, variables, names, and functions
  - But it doesn't include any *quantifiers*

- **Logical Axioms**

(T) All tautologies of TFL

(Ref) $x = x$

(Equiv) $x = y \rightarrow (\mathcal{A}(\ldots x \ldots x \ldots) \leftrightarrow \mathcal{A}(\ldots y \ldots x \ldots))$

- **Logical Inference Rules**

(MP) $\mathcal{A}, \mathcal{A} \rightarrow \mathcal{B} \vdash \mathcal{B}$

(VS) $\mathcal{A}(\chi) \vdash \mathcal{A}(a)$

(where χ is any variable, and a is any variable or name)

PRA: Successor

- PRA has two axioms governing the **successor function** (i.e. the function from one number to the next)
  (1) $Sx \neq 0$
  (2) $Sx = Sy \rightarrow x = y$

- PRA also includes a quantifier-free **induction rule**

(Ind) $\mathcal{A}(0), \mathcal{A}(x) \rightarrow \mathcal{A}(Sx) \vdash \mathcal{A}(y)$

PRA: Primitive Recursive Functions

- PRA allows you to lay down the defining equations for any **primitive recursive** function you like
- Roughly, a primitive recursive function is a function that can be computed using only "for"-loops, i.e. loops which have a pre-specified finite limit on the number of iterations
- Formally, we specify some basic PR-functions, and then lay down some operations which map PR-functions to PR-functions

Basic PR-Functions

- **The successor function:** Sx
- **Constant functions:** $C^k_n(x_1, \ldots, x_k) = n$
- **Projection functions:** $P^k_n(x_1, \ldots, x_k) = x_n$
  (where $1 \leq n \leq k$)

Composition

- Let f be an n-adic PR-function, and $g_1, \ldots, g_n$ be m-adic PR-functions
- We can make a new PR-function by **composing** f with $g_1, \ldots, g_n$, which we write as $(f \circ (g_1, \ldots, g_n))$
- **Formal definition:** $(f \circ (g_1, \ldots, g_n))(x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$

Primitive Recursion

- Let g be an n-adic PR-function, and h be an $(n+2)$-adic PR-function
- We can define a new $(n+1)$-adic PR-function, f, by **primitive recursion** on g and h
- We lay down a base clause, and a recursion clause:

(Base) $f(0, x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$

(Recursion) $f(Sy, x_1, \ldots, x_n) = h(y, f(y, x_1, \ldots, x_n), x_1, \ldots, x_n)$

Two Examples of Primitive Recursion

- **Addition**

(Base) $x + 0 = x$

(Recursion) $x + Sy = S(x + y)$

- **Multiplication**

(Base) $x \times 0 = 0$

(Recursion) $x \times Sy = (x \times y) + x$

Why Identify Finitary Mathematics with PRA?

- It is very plausible that PRA should count as finitary
  - There are no unbounded quantifiers
  - All universal generalisations are schematic
  - Primitive recursive functions are guaranteed to be computable in a finite period of time

- Some formalists have suggested that **general recursive functions** are finitistic, but that is controversial
  - General recursive functions are functions that can be computed with "until"-loops, which do not have a pre-specified limit to their iterations
  - So there is no guarantee that a general recursive function can be computed in a finite period of time

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The Limits of Finitary Mathematics

- **Finitary Mathematics** makes up a **very small** fragment of mathematics as it is practised today
- It is expressively impoverished: you cannot negate a universal generalisation!
- This is where **Ideal Mathematics** comes in. . .

Ideal Elements

- Mathematicians have often introduced 'ideal' elements to make their theories easier to use
- These ideal elements are not the mathematicians' primary interest, but positing them makes it much easier to talk and think about *other* things
- Two examples:
  - **Ideal points at infinity in geometry:** these points make it easier to talk about the relations between "real" lines
  - **Imaginary numbers (e.g.$\sqrt{-1}$):** these numbers make it easier to talk about the roots of equations

Ideal Mathematics

- Hilbert went one step further, and said that **all mathematics** which goes beyond the finitary is ideal
- Hilbert accepts all of classical, non-finitary mathematics, but only as a *meaningless game* we play with symbols
- The point of playing this game is that it helps us to prove meaningful results in finitary mathematics

Consistency

- Of course, there are limits on what kinds of ideal games we are allowed to play
- At a bare minimum, **Ideal Mathematics** (IM) must never deliver results that are refuted by **Finitary Mathematics** (FM)
- In other words: IM must be **consistent** with FM
  - $\mathrm{IM} + \mathrm{FM} \nvdash 0 = 1$
- Since IM includes FM, we just require that IM be consistent
  - $\mathrm{IM} \nvdash 0 = 1$

Conservativeness

- We might also want to impose the (apparently) stronger requirement that IM never deliver any results that are not already certified by FM
- In other words: IM must be **conservative** over FM
  - If $\text{IM} + \text{FM} \vdash \mathcal{A}$, then $\text{FM} \vdash \mathcal{A}$
    (where $\mathcal{A}$ is any formula in the language of FM)
- IM never lets us prove **new** finitary results
- IM only ever makes it **easier** to prove old finitary results, that we could already prove in FM

Back to Consistency

- We can immediately step from *consistency* to *conservativeness*, if we assume that FM is **negation-complete**
  - FM is negation-complete iff, for each finitary sentence $\mathcal{A}$, FM $\vdash \mathcal{A}$ or FM $\vdash \neg\mathcal{A}$
- Given this assumption, the requirement that IM be **conservative** over FM just becomes the requirement that IM be **consistent** with FM
  - Suppose that IM + FM $\vdash \mathcal{A}$, but FM $\nvdash \mathcal{A}$. If FM is negation complete, then FM $\vdash \neg\mathcal{A}$. So IM + FM is inconsistent.
- Unfortunately, PRA is **not** negation-complete...

The Requirement of a Consistency Proof

There is just one condition, albeit an absolutely necessary one, connected with the method of ideal elements. That condition is a proof of consistency, for the extension of the domain by the addition of ideal elements is legitimate only if the extension does not cause contradictions to appear in the old, narrower domain, or, in other words, only if the relations that obtain among the old structures when the ideal structures are deleted are always valid in the old domain.

(Hilbert 'On the infinite', in B&P, p.199)

Proving Consistency

- We can explain what it means for IM + FM to be consistent in **finitary** terms

- Formal proofs are finite sequences of symbols, and the claim that IM + FM is consistent is a finitary claim about those finitary proofs:
  - No proof which starts from axioms of IM + FM ends with '$0 = 1$'

- **But importantly:** the proof that IM + FM is consistent *must* be given in *finitary* terms
  - It would be illegitimate to give an infinitary (i.e. ideal) proof, because the whole point of the proof is to establish that infinitary methods are legitimate!

Gödel's Incompleteness Results

- Hilbert was confident that he could give a finitary proof of the consistency of ideal mathematics
- Hilbert and his school did make impressive progress on this front
- But in 1931, Kurt Gödel published his Incompleteness Theorems
- The Second Incompleteness Theorem implies that, if IM is consistent, then you cannot prove that IM + FM is consistent within IM, let alone within FM
- More on Gödel's Incompleteness Theorems next week!

For the Seminar

- Please read the following two items:
  - Hilbert 'On the infinite', in Benacerraf and Putnam *Philosophy of Mathematics: Selected Readings*
  - Giaquinto *The Search for Certainty*, Part IV Chapters 3 and 4
- You can find the Giaquinto chapters on the VLE, as well as information about where to find the Hilbert article

For the Next Lecture

- We are going to be discussing the fundamentals of Gödel's Incompleteness Theorems
- Please read the following in advance:
  - Nagel and Newman (2001) *Gödel's Proof*, ch. 7