The Foundations of Mathematics
Lecture Two

Russell's Paradox and his Logicism

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Russell's Paradox and his Logicism

Frege's Road Not Taken

(HP) $NxFx = NxGx \leftrightarrow F \approx G$

- Two important points about (HP):
  (i) $F \approx G$ can be given a purely logical definition
  (ii) Then, using nothing but (HP) and logic, we can derive all of the axioms of arithmetic (**Frege's Theorem**)

- This makes it tempting to treat (HP) as a kind of definition of number

- But Frege rejected this idea, because of the **Julius Caesar Problem**
  - We cannot use (HP) to determine whether Julius Caesar $= Nx(x \neq x)$

Frege's Explicit Definition

- Frege reacted by introducing his *extensions*, which we are calling **classes**
- For Frege in *Grundlagen*, numbers are classes of properties:
  - $NxFx = \{G : G \approx F\}$
- Frege told us very little about classes in *Grundlagen*
- For more, we have to turn to his more technical work: *Die Grundgesetze der Arithmetik*

The *Grundgesetze* Definition of Number

- In Grundgesetze, Frege subtly changed his definition of number
- Rather than defining numbers as classes of **properties**, Frege defined them as classes of **classes**
- Roughly, the number of Fs is the class of all **classes** that are equinumerous to F

$$NxFx = \{x : \exists G(x = \{y : Gy\} \wedge G \approx F)\}$$

- This was helpful because, in Frege's system, classes of **properties** would be of different type from classes of objects; but classes of **classes** are just classes of (abstract) objects

The *Grundgesetze* Definition of Number

- If all the symbols in Frege's definition of the numbers were a bit bamboozling, here is a simpler way of putting exactly the same thing:
  - The number 1 = the class of all classes with 1 member
  - The number 2 = the class of all classes with 2 members
  - The number 3 = the class of all classes with 3 members
  - ...
- In general, the number n is the class of all classes with n members

Basic Law V

- Frege had an axiom governing classes of objects, which he called **Basic Law V**:

  (V) $\{x : Fx\} = \{x : Gx\} \leftrightarrow \forall x(Fx \leftrightarrow Gx)$

  In words: the class of Fs = the class of Gs iff every F is a G and every G is an F

- Frege obviously regarded (V) as a logical law of some kind, but it is not clear why

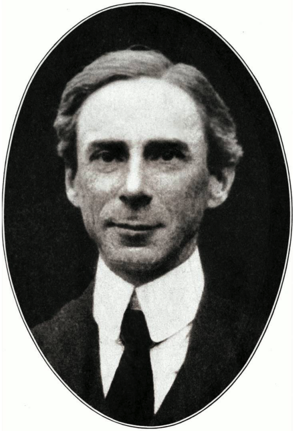
- We still face a version of the **Julius Caesar Problem**
  - We cannot use (V) to determine whether Julius Caesar $= \{x : x \neq x\}$

- But much more importantly, we now know that (V) *cannot* be a logical law, because it is **inconsistent**

Russell's Paradox and his Logicism

Introducing Russell's Paradox

- Russell first presented his paradox to Frege in a letter in 1902
- The second volume of Frege's *Grundgesetze* was already at the press
- Frege saw the trouble straight away, and although he tried to fix (V), he eventually gave up on his whole logicist project



Proving that Classes Exist

- Last week we saw that we could use (HP) to prove that $NxFx$ exists, no matter what property F is

- In exactly the same way, we can use (V) to prove that $\{x : Fx\}$ exists, no matter what property F is

  1. $\{x : Fx\} = \{x : Fx\} \leftrightarrow \forall x(Fx \leftrightarrow Fx)$ [V]
  2. $\forall x(Fx \leftrightarrow Fx)$ [pure logic]
  3. $\therefore\ \{x : Fx\} = \{x : Fx\}$ [$\leftrightarrow$E, 1,2]
  4. $\therefore\ \exists y(y = \{x : Fx\})$ [free $\exists$I, 3]

- In modern terminology, (V) entails a principle called **Naïve Comprehension**:

(NC) For any property F, $\{x : Fx\}$ exists

Classes that are not Members of Themselves

- Classes have members: $x \in \{y : Fy\} \leftrightarrow Fx$

  $\in$ is the modern symbol for the membership relation

- In many cases, the members of classes are not themselves classes, they are ordinary objects like you and me
  - The class of humans does not have any classes as members, it has humans as members

- But sometimes classes can have classes as members
  - The class of classes with fewer than 9 members has other classes for members

- And on the face of it, it seems that sometimes, classes can even be members of **themselves**
  - The class of all classes is a member of itself

The Russell Class

- Now consider the class of all classes that are **not** members of themselves, and call it R
  - $R =_{df} \{x : x \notin x\}$
- We can use (NC) to prove that R exists, and at first, R might seem like a perfectly good class
  - The class of people is a member of R, because the class of people is not a member of the class of people
  - The class of all the classes is not a member of R, because the class of all classes is a member of the class of all classes
- But now ask: $R \in R$?

Russell's Paradox

- $\forall x(x \in R \leftrightarrow x \notin x)$
- $R \in R \leftrightarrow R \notin R$
  - In words: R is a member of R if and only if R is not a member of R
- **Contradiction!!!**

A Helpful Parallel

If all this talk of classes which are not members of themselves is confusing you, then it might be helpful to look at the following analogous paradox...

The Barber Paradox

- The Barber shaves **only** the people in his village who do not shave themselves
- The Barber shaves **all** the people in his village who do not shave themselves
- Does The Barber shave himself?

The Barber Paradox

- If The Barber shaves himself, then he **does not** shave himself
- If The Barber does not shave himself, then he **does** shave himself
- So The Barber shaves himself iff he does not shave himself!

Russell's Paradox and the Barber

- The Barber Paradox just shows that there couldn't be a barber who shaved all and only the people who do not shave themselves

- Couldn't we also just say that Russell's Paradox just shows that there is no such class as R?

- Things are not that easy, because the existence of R is guaranteed by (NC):

  (NC) For any property F, $\{x : Fx\}$ exists

- And (NC) follows from Frege's (V):

  (V) $\{x : Fx\} = \{x : Gx\} \leftrightarrow \forall x(Fx \leftrightarrow Gx)$

- So Russell's Paradox shows that (V) is **inconsistent**

Frege's Reply to Russell

Your discovery of the contradiction has surprised me beyond words and, I should almost like to say, left me thunderstruck, because it has rocked the ground on which I meant to build arithmetic. [...] It is all the more serious as the collapse of my law V seems to undermine not only the foundation of my arithmetic but the only possible foundations of arithmetic as such [...]

Frege's Reply to Russell

The second volume of my Grundgesetze is to appear shortly. I shall have to give it an appendix where I will do justice to your discovery. If only I could find the right way of looking at it!

(Frege to Russell, 22nd June 1902, reprinted in Beaney 1997: 254)

Russell's Paradox and his Logicism

Russell the Logicist

- Although he wrecked Frege's logicist project, Russell was actually a logicist himself
- And in fact, he thought that Frege had got a lot of things right
- For example, like Frege, Russell wanted to define the number n as the class of all classes with n members

Russell on Frege

The question "What is a number?" is one which has often been asked but has only been correctly answered in our own time. The answer was given by Frege in 1884, in his Grundlagen der Arithmetik. Although this book is quite short, not difficult, and of the very highest importance, it attracted almost no attention, and the definition of number which it contains remained practically unknown until it was rediscovered by the present author.

(Bertrand Russell, Introduction to Mathematical Philosophy, reprinted in Benacerraf & Putnam 1983: 167)

Russell's Task

- Russell's task was to find a way of eliminating his paradox from the theory of classes
- He experimented with very many ways of trying to do this, but settled on one by the time of his great work *Principia Mathematica* (written with his old teacher, Alfred Whitehead)
- Russell introduced **type theory** as a way of blocking his paradox
- The details of Russell's own type theory are highly complex, and so we will focus on the basics today

(I will briefly mention where I have simplified things later)

Classes of Classes

- Russell's Paradox would have never got going if you were not allowed to take classes **of classes**
- That might make you tempted to deal with the paradox by simply banning us from taking classes of classes
- Unfortunately, that would force us to give up Frege's definition of numbers:
  - The number 2 $=_{df}$ the class **of all classes** with 2 members
- So what we need to do is find a way of introducing classes of classes in a controlled, careful way

Type Theory

- In Russell's type theory, every entity has a **type**
  - Ordinary individuals *which are not classes* are **type 0**
  - Classes of individuals are **type 1**
  - Classes of classes of individuals are **type 2**
  - Classes of classes of classes of individuals are **type 3**
  - ...
- In general, an entity of type n can only be a member of a class of type $n+1$
- **But more than that:** if a is of type n and b is not of type $n+1$, then '$a \in b$' is not merely false, it is **meaningless**

Some Examples

- Frege is not a class, so he is type 0
- $\{x : x \text{ is human}\}$ is a class of individuals, so it is type 1
- Therefore, 'Frege $\in \{x : x \text{ is human}\}$' is perfectly **meaningful** (and in fact *true*)

Some Examples

- Frege is type 0
- $\{x : x \text{ is a dog}\}$ is type 1
- Therefore, 'Frege $\in \{x : x \text{ is a dog}\}$' is also **meaningful** (although *false*)

Some Examples

- $\{x : x \text{ is human}\}$ is type 1
- $\{x : x \text{ is a dog}\}$ is type 1
- So, '$\{x : x \text{ is human}\} \in \{x : x \text{ is a dog}\}$' is **meaningless**
  - It is not true
  - It is not even false
  - It is just a meaningless collection of symbols, like 'Fribble frabble dibble dabble'

Blocking Russell's Paradox

- In general, '$a \in a$' is always **meaningless**
- This blocks Russell's Paradox
- We can no longer ask whether $R \in R$, because '$R \in R$' is just meaningless!
- How much of (V) can we save?

What is Left of (V)?

- We can have a version of (V) for each type:
  - **Informally:** If *a* and *b* are two entities of type $n+1$, then $a = b$ iff every type n entity that is a member of *a* is a member of *b* too, and *vice versa*

(V^n) $\{x^n : F^{n+1}x^n\} = \{x^n : G^{n+1}x^n\} \leftrightarrow \forall x^n(F^{n+1}x^n \leftrightarrow G^{n+1}x^n)$

The variable x^n ranges over entities of type n, and F^{n+1} and G^{n+1} are properties which can be meaningfully applied to entities of type n

- We will have a different version of (V^n) for each different type, and in general (V^n) will tell us how classes of type $n+1$ behave
  - (V^0) will tell us how type 1 classes behave
  - (V^1) will tell us how type 2 classes behave
  - ...

Russell's Paradox and his Logicism

The No-Class Theory

- Despite offering a theory of classes, Russell didn't **really** believe in classes
- He thought that class-talk was a convenient way of speaking, but that it could all be analysed away in favour of what he called **propositional functions**

Propositional Functions

- The notion of a propositional function is a bit obscure, but roughly, they are what you get when you replace something in a proposition with a variable
  - **Proposition:** Socrates is wise
  - **Propositional function:** $\hat{x}$ is wise

    (But are propositional functions meant to be symbols? Or are they really properties represented by symbols?)

- The idea is that we could re-write any claim about classes as a claim involving only propositional functions:
  - $F(\{x : \phi x\}) \leftrightarrow_{df} \exists\psi(\forall x(\phi x \leftrightarrow \psi x) \wedge F(\psi\hat{x}))$
  - Socrates $\in \{x : x \text{ is human}\} \leftrightarrow_{df}$ Socrates is human

Type Restrictions Explained

- When we remember that Russell's (mature) view was that there are no classes, type theory can actually make a bit more sense
- Socrates $\in \{x : x \text{ is human}\} \leftrightarrow_{df}$ Socrates is human
- $\{x : x \text{ is human}\} \in \{x : x \text{ is human}\} \leftrightarrow_{df}$???
  - **Ungrammatical:** $\{x : x \text{ is human}\} \in \{x : x \text{ is human}\} \leftrightarrow$ is human is human

Russell's Paradox and his Logicism

Simple vs. Ramified Theories of Types

- As well as splitting classes up into types, Russell split **propositions** into types too
- Propositions were to come in a series of orders, depending on the types of quantifier they contain
- The kind of type theory which I have presented is known as **simple** type theory, whereas a type theory which combines the typing of classes with the typing of propositions is known as **ramified** type theory

A Sketch of Ramification

- Propositions come in **orders**
  - **Order 0** propositions do not contain any quantifiers
    (Example: Socrates is wise)
  - **Order 1** propositions quantify over order 0 propositions (but no higher)
    (Example: every order 0 proposition is true or false)
  - **Order 2** propositions quantify over order 1 propositions (but no higher)
    (Example: I believe some order 1 proposition)
  - ...
- You can only ever quantify over one order of propositions with a single variable!!!

Ramification and the Liar Paradox

- **The Liar Paradox**:
  - If I say 'I am lying', then what I have said is true iff it is false
- **The Ramified Solution**
  - 'I am lying' means *There is a proposition which I am asserting, but which is false*
  - But you can only quantify over one order at a time
  - So 'I am lying' must mean *There is an order n proposition which I am asserting, but which is false*
  - And now 'I am lying' is a proposition of order $n+1$
  - So 'I am lying' is just plain false: I am not currently asserting an order n proposition!

The Vicious Circle Principle

- Russell didn't just motivate ramified type theory by showing that it blocks paradoxes, he tried to motivate it with his **Vicious Circle Principle**

(VCP) "Whatever involves all of a collection must not be one of the collection" (Russell 1908: 63)

- The idea is that when we are defining something (say a particular proposition), we must not in any way mention that thing in the definition
  - We must not directly refer to it
  - We must not quantify over it
- 'I am asserting some proposition which is false' defines a proposition by quantifying over a totality that includes the very proposition you are trying to define!

Keep It Simple, Stupid!

- Ramification turns out to be *very* complicated
- It also makes the type theory a lot weaker
  - There are lots of things you can prove in a simple theory of types, but not in a ramified theory
- Russell tried to get around this by introducing an Axiom of Reducibility, which was meant to give type theory it's power back without re-introducing paradoxes
  - For a sketch of how this is meant to work, see *Shapiro* pp.120–3
- But it is unclear what justifies the Axiom of Reducibility
  - For classic discussion, see Ramsey's 'Foundations of Mathematics'

Russell's Paradox and his Logicism

Logicism Vindicated?

- It might seem like Russell has vindicated logicism
- Frege showed us how to derive arithmetic from the theory of classes (which Frege thought of as a branch of logic), and Russell has shown us how to eliminate his paradox
- But, as ever, things are not quite so simple...

Frege's Definition

- Frege defined the number n as the class of all classes with n members
- For example, $2 =_{df}$ the class of all classes with 2 members
- Russell wanted to give this definition too, but he couldn't leave it quite as it is...

Types of Number

- For Russell, there is no such thing as the class of **all classes** with 2 members
- Such a class cannot appear anywhere in his type theory
- Instead, you will have different number 2s for all the different types of entity:
  - The class of all **classes of individuals** with 2 members
  - The class of all **classes of classes of individuals** with 2 members
  - ...

What if there are only Finitely many Individuals?

- Suppose there are only 2 individuals
- Now consider the numbers 3 and 4 for individuals:
  - The class of all **classes of individuals** with 3 members
  - The class of all **classes of individuals** with 4 members
- Because there are only 2 individuals, there are no classes of individuals with 3 or 4 members
- So the class of all classes of individuals with 3 members will be empty, and so will the class of all classes of individuals with 4 members
- Two classes (of the same type) are identical if they have the same members, and so it follows that, for individuals, 3=4!

Moving up the Types

- Given appropriate background assumptions, you can prove that if there are n individuals, there are 2^n classes of individuals
- So, for classes of individuals, $3 \neq 4$!
- But, for classes of individuals, $5 = 6$!
- We can go up the hierarchy again and again, but if we started with just finitely many individuals, we'll keep getting false identities between (bigger and bigger) numbers

The Axiom of Infinity

- To get around this problem, Russell had to add an **Axiom of Infinity** to his theory:

  (Inf) There are infinitely many individuals

- But this axiom hardly looks like an axiom of **logic**

- It is true that we can formulate (Inf) using nothing but concepts belonging to logic, but does that automatically make (Inf) a logical principle?

Frege vs. Russell

- It is important to contrast Frege's logicism with Russell's here
- Frege did not **assume** that there were infinitely many objects (or individuals); he **proved** it from (HP)
- The reason he could do this was that (HP) is **impredicative**:
  - The numbers introduced on the left-hand-side of (HP) are quantified over on the right-hand-side
- This impredicativity means that (HP) doesn't just govern numbers of individuals, but numbers of numbers as well, which is how Frege demonstrated that there were infinitely many numbers

Frege vs. Russell

- Frege also took (V) to be impredicative in exactly the same way:
  - The classes introduced on the left-hand-side of (V) are quantified over on the right-hand-side
- This impredicativity means that (V) doesn't just govern classes of individuals, but classes of classes as well
- But unfortunately, it was this impredicativity which led to Russell's Paradox!

Frege vs. Russell

- In Russell's type theory, we do not have Frege's impredicative (V)
- Rather than one principle governing classes of individuals **and** classes of classes
  - We have (V^0) for classes of individuals
  - We have (V^1) for classes of classes of individuals
  - ...
- This saves us from Russell's Paradox, but makes it impossible to **prove** that there are infinitely many things; Russell just has to **assume** that they are with his axiom (Inf)

Frege vs. Russell

- So at this point, logicists seem to be faced with a dilemma:

(1) Follow Frege, and use an impredicative version of (V)

- **Pro:** You can prove that there are infinitely many things
- **Con:** The system is vulnerable to Russell's Paradox

(2) Follow Russell, and introduce his theory of types

- **Pro:** The system is immune to Russell's Paradox
- **Con:** You cannot *prove* that there are infinitely many things, you just have to *assume* it

The End of Logicism?

- Is this the end of logicism?
- No!
  - Ramsey (1924) tried to save Russell's logicism by combining it with the philosophical doctrines of Wittgenstein's *Tractatus*
  - Hale and Wright (2001) have tried to resurrect Frege's logicism by working just with (HP), not (V)
- You can find out more about both programmes in the Reading List

What Happens to Classes?

- Classes (now known as **sets**) have remained a fundamental part of mathematics
- The modern solution to Russell's paradox is a bit similar to Russell's type theory
  - The **cumulative iterative conception** of sets pictures sets as coming in levels, and sets can only have members from lower levels
- But there are important **differences** too
  - A *rank n* set can have anything of rank $0 \leq m < n$ as a member
  - When set a has greater or equal rank than b, '$a \in b$' is false, not meaningless
  - The variables in modern set theory are untyped, so a single variable can have all sets of all ranks as values

What Happens to Classes?

- Most contemporary philosophers deny that modern set theory is a branch of **logic**
- Received wisdom is that the structure described by set theory is too **mathematically rich** to be extracted out of pure logic
- For a technically advanced discussion of the difference between set theory and type theory, see: Button and Trueman (forthcoming), 'Against cumulative type theory', *Review of Symbolic Logic*

Seminar

- For the seminar, please read:
  - Bertrand Russell, 'Selections from *Introduction to Mathematical Philosophy*' in Putnam and Benacerraf (1983)
  - Giaquinto (2002) Part II chs 3 & 4
- Both of these readings can be found on the VLE

References

- Beaney, M (1997) *The Frege Reader* (Blackwell)
- Benacerraf and Putnam (1983) *Philosophy of Mathematics: Selected Readings* (CUP)
- Hale and Wright (2001) *The Reason's Proper Study* (OUP)
- Ramsey (1924) 'The Foundations of Mathematics', reprinted in Braithwaite eds (1931) *The Foundations of Mathematics and Other Logical Essays* (Routledge)