Intermediate Logic Spring
Lecture Two

Possible World Semantics

Rob Trueman
rob.trueman@york.ac.uk

University of York

Possible World Semantics

K = TFL + MP + Nec

$$
\begin{array}{l|ll}
m & \Box(\mathcal{A} \to \mathcal{B}) & \\
n & \Box\mathcal{A} & \\
 & \Box\mathcal{B} & \text{MP, } m,
\end{array}
$$

$$
\begin{array}{l|l|ll}
m & & \\
\cline{3-3}
n & & \mathcal{A} & \\
 & \multicolumn{2}{l}{\Box\mathcal{A}} & \text{Nec, } m\text{–}n
\end{array}
$$

No line above line *m* may be cited by any rule within the subproof begun at line *m*

T = K + the T Rule

$$\begin{array}{l|ll} m & \Box\mathcal{A} & \\ & \mathcal{A} & \text{T, } m \end{array}$$

S4 = T + the S4 Rule

m	$\Box\mathcal{A}$	
	$\Box\Box\mathcal{A}$	S4, m

S5 = T + the S5 Rule

m	$\Diamond\mathcal{A}$	
	$\Box\Diamond\mathcal{A}$	S5, m

This Week: Semantics

- This week, we will look at the **semantics** for Modal Logic (ML)
- A semantics for a language is a method for assigning truth-values to the sentences in that language
- So a semantics for ML is a method for assigning truth-values to the sentences of ML

The Big Idea

- A sentence is not just true or false, *full stop*
- A sentence is true or false, *at a given possible world*
  - One sentence can be true at some worlds, false at others
- $\Box\mathcal{A}$ means that $\mathcal{A}$ is true **at all possible worlds**
- $\Diamond\mathcal{A}$ means that $\mathcal{A}$ is true **at some possible world**

Possible World Semantics

Possible Worlds

- The first thing you need to include in an interpretation is a collection of *possible worlds*
- **What is a possible world!?**
- Intuitive Answer: A possible world is another way that this world could have been
- Official Answer: For now, it just doesn't matter!
  - As far as the formal logic goes, the possible worlds can be anything you like
  - All that matters is that you supply each interpretation with a non-empty collection of things labelled POSSIBLE WORLDS

Introducing Valuation Functions

- Once you have chosen your collection of possible worlds, you need to find some way of determining which sentences are true at which possible worlds
- To do that, we need to introduce the notion of a **valuation function**
- But before we can explain what a *valuation* function is, we need to talk about what *functions* in general are

Functions

- A function is a mathematical entity which maps **arguments** to **values**
- Here are some examples:

$x + 1$

1	2
589	590
1,003	1,004

x^2

2	4
5	25
10	100

$x \times y$

4	3	12
12	11	132
25	5	125

Back to Valuation Functions

- A **valuation function** for ML takes in a *sentence* and a *world* as its arguments, and returns a *truth-value* as its value
  - We can use numbers to represent the truth-values: 0 represents falsehood, 1 represents truth
- So if ν is a valuation function and w is a possible world, $\nu_w(\mathcal{A})$ is whatever truth-value ν maps $\mathcal{A}$ and w to
  - If $\nu_w(\mathcal{A}) = 0$, then $\mathcal{A}$ is false at world w on valuation ν
  - If $\nu_w(\mathcal{A}) = 1$, then $\mathcal{A}$ is true at world w on valuation ν

Atomic versus Complex

- Valuation functions are allowed to map any **atomic** sentence to any truth-value at any world
- But there are rules about which truth-values more complex sentences get assigned to at a world
- We'll start with the rules for the connectives from TFL

Semantic Rules for the Truth-Functional Connectives

(1) $\nu_w(\neg\mathcal{A}) = 1$ iff: $\nu_w(\mathcal{A}) = 0$

(2) $\nu_w(\mathcal{A} \wedge \mathcal{B}) = 1$ iff: $\nu_w(\mathcal{A}) = 1$ and $\nu_w(\mathcal{B}) = 1$

(3) $\nu_w(\mathcal{A} \vee \mathcal{B}) = 1$ iff: $\nu_w(\mathcal{A}) = 1$ or $\nu_w(\mathcal{B}) = 1$, or both

(4) $\nu_w(\mathcal{A} \rightarrow \mathcal{B}) = 1$ iff: $\nu_w(\mathcal{A}) = 0$ or $\nu_w(\mathcal{B}) = 1$, or both

(5) $\nu_w(\mathcal{A} \leftrightarrow \mathcal{B}) = 1$ iff: $\nu_w(\mathcal{A}) = \nu_w(\mathcal{B})$

What about the Modalities?

- Here are the obvious semantic rules to give for $\Box$ and $\Diamond$
  - $\nu_{w_1}(\Box\mathcal{A}) = 1$ iff $\forall w_2(\nu_{w_2}(\mathcal{A}) = 1)$
  - $\nu_{w_1}(\Diamond\mathcal{A}) = 1$ iff $\exists w_2(\nu_{w_2}(\mathcal{A}) = 1)$
- However, while these rules are nice and simple, they turn out not to be quite as useful as we would like
- As I mentioned last week, ML is meant to be a general framework for dealing with lots of different kinds of necessity
- As a result, we need our semantic rules for $\Box$ and $\Diamond$ to be a bit more flexible

Accessibility Relations

- An **accessibility relation**, R, is a relation between possible worlds
  - When Rw_1w_2, we say that w_1 *accesses* w_2
- Roughly, to say that w_1 accesses w_2 is to say that w_2 is possible *relative to* w_1
- By introducing accessibility relations, we open up the idea that a given world might be possible *relative to* some worlds, but not others
- This turns out to be a **very** fruitful idea when studying different modal systems

Semantic Rules for the Modalities

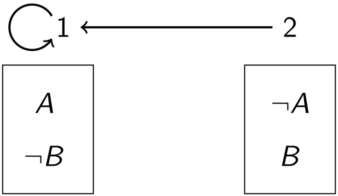
(6) $\nu_{w_1}(\Box\mathcal{A}) = 1$ iff $\forall w_2(Rw_1w_2 \rightarrow \nu_{w_2}(\mathcal{A}) = 1)$

(7) $\nu_{w_1}(\Diamond\mathcal{A}) = 1$ iff $\exists w_2(Rw_1w_2 \wedge \nu_{w_2}(\mathcal{A}) = 1)$

Interpretations Consist of 3 Things:

- *A collection of possible worlds, W*
  - W can really be a collection of **anything** you like
  - All that matters is that W be non-empty
- *An accessibility relation, R*
  - R is a relation between the members of W
  - For now, R can be **any** relation between the members of W you like
- *A valuation function, ν*
  - ν can map **any** atomic sentence to **any** truth-value at **any** world
  - But when it comes to more complex sentences, ν has to follow rules (1)–(7)

A Diagrammatic Example



- **True or False at 1?**
  - $B \to A$, $\Diamond A$, $\Diamond B$
- **True of False at 2?**
  - $B \to A$, $\Diamond \neg A$, $\Box \neg B$

Possible World Semantics

Semantic Concepts

- $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \therefore \mathcal{C}$ is **valid** iff there is no world in any interpretation at which $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n$ are all true and $\mathcal{C}$ is false
- $\mathcal{A}$ is a **logical truth** iff $\mathcal{A}$ is true at every world in every interpretation
- $\mathcal{A}$ is a **contradiction** iff $\mathcal{A}$ is false at every world in every interpretation.
- $\mathcal{A}$ is **consistent** iff $\mathcal{A}$ is true at some world in some interpretation

Soundness and Completeness Results

- As before, we will use $\vDash$ to express the logical consequence relation
- But we will also add a subscript, just like we did with $\vdash$:
  - $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \therefore \mathcal{C}$ is valid
  - $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vDash_{\mathsf{K}} \mathcal{C}$
- Why did we add the K subscript? Because of the following results:
  - **Soundness:** If $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vdash_{\mathsf{K}} \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vDash_{\mathsf{K}} \mathcal{C}$
  - **Completeness:** If $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vDash_{\mathsf{K}} \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vdash_{\mathsf{K}} \mathcal{C}$

For Proofs, see a Textbook!

- Stating these soundness and completeness results is one thing, *proving* them is another!
- We won't try to do that in this module, but you can find proofs of (similar) results in any of the following textbooks:
  - Garson's *Modal Logic for Philosophers*
  - Priest's *An Introduction to Non-Classical Logic*
  - Hughes and Cresswell's *A New Introduction to Modal Logic*

Possible World Semantics

What about the Stronger Modal Systems?

- Our definition of validity is sound and complete for K
- Where does that leave the more powerful modal systems we looked at last week: T, S4 and S5?
- Well, they are all **unsound**, relative to that definition of validity!
  - $\Box A \vdash_{\mathrm{T}} A$, but $\Box A \nvDash_{\mathrm{K}} A$
- Does that mean that these stronger systems are a waste of our time?
- Not at all! When dealing with modal systems stronger than K, we just need to tweak our definition of validity to fit

Reflexive Accessibility Relations

- When I introduced the idea of an *accessibility relation*, I said that it could be any relation between worlds you liked
  - That is how we were thinking of accessibility relations in our definition of $\vDash_K$
- But if we wanted, we could start putting restrictions on the accessibility relation
- For example, we might insist that it must be **reflexive**:
  - $\forall wRww$

A New Definition of Validity

- $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vDash_{\mathsf{T}} \mathcal{C}$ iff there is no world in any interpretation **which has a reflexive accessibility relation**, at which $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n$ are all true and $\mathcal{C}$ is false

- It turns out that T is sound and complete relative to this new definition of validity
  - **Soundness:** If $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vdash_{\mathsf{T}} \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vDash_{\mathsf{T}} \mathcal{C}$
  - **Completeness:** If $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vDash_{\mathsf{T}} \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vdash_{\mathsf{T}} \mathcal{C}$

Validating the T Rule

- If you want proofs of these results, you should look at the textbooks I mentioned earlier

- However, it is relatively easy to see how insisting that the accessibility relation must be reflexive will vindicate the T rule

$$\begin{array}{l|ll} m & \Box\mathcal{A} & \\ & \mathcal{A} & \quad\text{T}, m \end{array}$$

- To see this, imagine trying to cook up a counter-interpretation to this: $\Box\mathcal{A} \vDash_{\text{T}} \mathcal{A}$

Validating the T Rule

- You would need to construct a world, w, at which $\Box\mathcal{A}$ was true, but $\mathcal{A}$ was false
- If $\Box\mathcal{A}$ is true at w, then $\mathcal{A}$ must be true at every world w accesses
- But since the accessibility relation is reflexive, w accesses w
- So $\mathcal{A}$ must be true at w
- But now $\mathcal{A}$ must be true *and* false at w!

Possible World Semantics

Transitive Accessibility Relations

- As well as requiring that our accessibility relation be reflexive, we might also require that it be **transitive**:
  - $\forall w_1 \forall w_2 \forall w_3((Rw_1 w_2 \wedge Rw_2 w_3) \rightarrow Rw_1 w_3)$

- $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n \vDash_{S4} \mathcal{C}$ iff there is no world in any interpretation **which has a reflexive and transitive accessibility relation**, at which $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n$ are all true and $\mathcal{C}$ is false

- It turns out that S4 is sound and complete relative to this new definition of validity
  - **Soundness:** If $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n \vdash_{S4} \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n \vDash_{S4} \mathcal{C}$
  - **Completeness:** If $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n \vDash_{S4} \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, ...\mathcal{A}_n \vdash_{S4} \mathcal{C}$

Validating the S4 Rule

- It is relatively easy to see how insisting that the accessibility relation must be reflexive and transitive will vindicate the S4 rule

$$\begin{array}{l|ll} m & \Box\mathcal{A} & \\ \hline & \Box\Box\mathcal{A} & \text{S4}, m \end{array}$$

- To see this, imagine trying to cook up a counter-interpretation to this: $\Box\mathcal{A} \vDash_{\text{S4}} \Box\Box\mathcal{A}$

Validating the S4 Rule

- You would need to construct a world, w_1, at which $\Box\mathcal{A}$ was true, but $\Box\Box\mathcal{A}$ was false
- If $\Box\Box\mathcal{A}$ is false at w_1, then w_1 must access some world, w_2, at which $\Box\mathcal{A}$ is false
- Equally, if $\Box\mathcal{A}$ is false at w_2, then w_2 must access some world, w_3, at which $\mathcal{A}$ is false
- We just said that w_1 accesses w_2, and w_2 accesses w_3; so since the accessibility relation is transitive, w_1 must access w_3
- Since $\Box\mathcal{A}$ is true at w_1, and w_3 is accessible from w_1, it follows that $\mathcal{A}$ must be true at w_3
- So $\mathcal{A}$ is true *and* false at w_3!

Possible World Semantics

An Equivalence Relation for an Accessibility Relation

- As well as requiring that our accessibility relation be reflexive and transitive, we might also require that it be **symmetric**:
  - $\forall w_1 \forall w_2 (Rw_1w_2 \rightarrow Rw_2w_1)$
- Logicians call relations which are reflexive, symmetric and transitive, **equivalence relations**

Another Definition of Validity

- $\mathcal{A}_1, \mathcal{A}_2, \ldots\mathcal{A}_n \vDash_{S5} \mathcal{C}$ iff there is no world in any interpretation **whose accessibility relation is an equivalence relation**, at which $\mathcal{A}_1, \mathcal{A}_2, \ldots\mathcal{A}_n$ are all true and $\mathcal{C}$ is false

- It turns out that S5 is sound and complete relative to this new definition of validity
  - **Soundness:** If $\mathcal{A}_1, \mathcal{A}_2, \ldots\mathcal{A}_n \vdash_{S5} \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots\mathcal{A}_n \vDash_{S5} \mathcal{C}$
  - **Completeness:** If $\mathcal{A}_1, \mathcal{A}_2, \ldots\mathcal{A}_n \vDash_{S5} \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots\mathcal{A}_n \vdash_{S5} \mathcal{C}$

Validating the S5 Rule

- It is relatively easy to see how insisting that the accessibility relation must be an equivalence relation will vindicate the S5 rule

$$\begin{array}{r|ll} m & \Diamond\mathcal{A} & \\ \hline & \Box\Diamond\mathcal{A} & \quad \text{S5, } m \end{array}$$

- To see this, imagine trying to cook up a counter-interpretation to this: $\Diamond\mathcal{A} \vDash_{\text{S5}} \Box\Diamond\mathcal{A}$

Validating the S5 Rule

- You would need to construct a world, w_1, at which $\Diamond\mathcal{A}$ was true, but $\Box\Diamond\mathcal{A}$ was false
- If $\Diamond\mathcal{A}$ is true at w_1, then w_1 must access some world, w_2, at which $\mathcal{A}$ is true
- Equally, if $\Box\Diamond\mathcal{A}$ is false at w_1, then w_1 must access some world, w_3, at which $\Diamond\mathcal{A}$ is false
- Since the accessibility relation is symmetric, we can infer that w_3 accesses w_1
- Thus, w_3 accesses w_1, and w_1 accesses w_2, and since the accessibility relation is also transitive, we can infer that w_3 accesses w_2
- But earlier we said that $\Diamond\mathcal{A}$ is false at w_3, which implies that $\mathcal{A}$ is false at every world which w_3 accesses
- So $\mathcal{A}$ is true *and* false at w_2!

A Universal Accessibility Relation

- In the definition of $\vDash_{S5}$, we stipulated that the accessibility relation must be an equivalence relation
- But it turns out that there is another way of getting a notion of validity fit for S5
- Rather than stipulating that the accessibility relation be an equivalence relation, we can instead stipulate that it be a **universal** relation
  - $\forall w_1 \forall w_2 R w_1 w_2$

One Last Definition of Validity

- $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vDash_{S5} \mathcal{C}$ iff there is no world in any interpretation **which has a universal accessibility relation**, at which $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n$ are all true and $\mathcal{C}$ is false

- It turns out that S5 is sound and complete relative to this alternatived definition of $\vDash_{S5}$
  - **Soundness:** If $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vdash_{S5} \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vDash_{S5} \mathcal{C}$
  - **Completeness:** If $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vDash_{S5} \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n \vdash_{S5} \mathcal{C}$

What does this Tell Us?

- These last results tell us that if we are dealing with a notion of necessity according to which **every** world is possible relative to **every** world, then we should use S5
- Most philosophers assume that the notions of necessity that they are most concerned with are of this kind
  - Logical necessity
  - Metaphysical necessity
- So S5 is the modal system that most philosophers use most of the time

Seminar 2

- The reading for Seminar 2 is:
  - *A Modal Logic Primer*, §4
- Please attempt at least some of the exercises before the seminar. (Why not meet up in groups to do the exercises together?)

Lecture & Seminar 3

- For next Lecture & Seminar 3, read:
  - David Lewis, *On the Plurality of Worlds*, ch.2 §§2.1–2.6
- Access to this chapter is available via the Reading List on the VLE
- A number of study questions will shortly be posted on the VLE